THE 7TH ANNUAL SPECIAL EDUCATION AND MATHEMATICS CONFERENCE

PREPARING FOR THE COMMON CORE STATE STANDARDS & ASSESSMENTS:

UPDATE 2015

A CONFERENCE FOR ALL GRADES K-12

WEDNESDAY, JANUARY 7, 2015

ST. PETERS UNIVERSITY

JERSEY CITY, NJ

TIME AND LOCATION: 12:00 P.M. – 1:00 P.M. – MCINTYRE B (100)

TITLE OF PRESENTATION: RICH PROBLEMS ILLUSTRATING THE PRACTICE – ATTEND TO PROBLEMS AND PERSEVERE IN SOLVING THEM

PRESENTER: JAY L. SCHIFFMAN

ROWAN UNIVERSITY

RICH PROBLEMS ILLUSTRATING THE PRACTICE – ATTEND TO PROBLEMS & PERSEVERE IN SOLVING THEM

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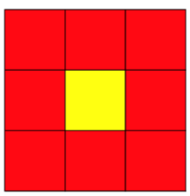
ABSTRACT: This workshop will focus on the mathematical practice of perseverance in problem solving. Problems will be selected from the branches of number, algebra and geometry illustrating this vital standard for mathematical practice alluded to in the Common Core.

SOME PROBLEMS AND DISSCUSSION ACTIVITIES:

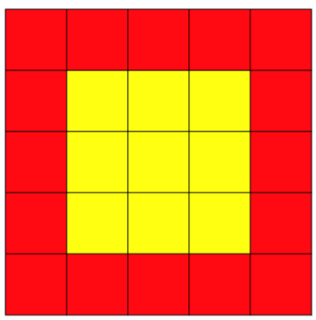
I. Use both inductive reasoning (five cases) and then deductive reasoning to solve the following number puzzle employing the given directives:
a. Pick any number.
b. Add 221 to the given selected number.
c. Multiply the sum by 2652.
d. Subtract 1326 from your product.
d. Divide your difference by 663.
e. Subtract 870 from your quotient.
f. Divide your difference by 4.
g. Subtract the original number from your quotient.

II. Consider the sum of two consecutive unit fractions with even denominators such as $\frac{1}{2}+\frac{1}{4}$. Do this for the first fifteen iterations. (i.e. Consider $\frac{1}{4}+\frac{1}{6}$, $\frac{1}{6}+\frac{1}{8}$, etc.) What do you notice when considering the numerators and denominators in each of these sums? Repeat this with the sum of two consecutive unit fractions with odd denominators such as $\frac{1}{3}+\frac{1}{5}$. Repeat for fifteen iterations. What do you notice about the numerators and denominators in the sums? Do you see any connections to geometry? If so, then explore such connections.

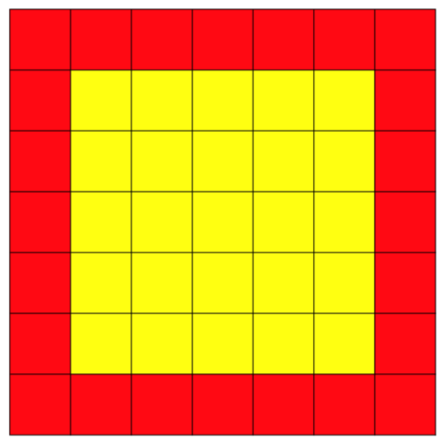
II. Here are the first three figures of a sequence formed by color tiles:



1st



2nd



3rd

(a). Find a pattern and describe the next two figures in the sequence.

(b). Describe the 100th figure. Include the number of each color of tile and the total number of tiles in the figure.

(c). Write algebraic expressions for the nth figure for (1) the number of yellow tiles, (2) the number of red tiles and (3) the total number of tiles.

IV. Determine the unit's digits for an integer to be a candidate for a perfect square as well as the last two digits. Based on your analysis, determine if the integers 742203, 113569, 90687529, 23853456 and 9354676 are possible candidates for perfect squares.

V. Consider the sequence 9, 98, 987, 9876, 98765, 987654,… and determine which integers in the sequence are divisible by each of the counting integers 2, 3, 4, 5, 6, 8, 9 and 10. Look for patterns. Finally determine if there are any primes in the sequence.

VI. Determine which of the following expressions is eventually larger: 2^n or n^{10}? Find the initial positive integer value for which this occurs. (Use appropriate tools strategically).

VII. Geometry and the Fibonacci sequence.

Consider any four consecutive terms in the Fibonacci sequence. First form the product of the first and fourth terms. Take twice the product of the second and third terms. Finally take the sum of the squares of the second and third terms in your sequence. Try to relate this to a theorem in plane geometry, conjecture based on several examples, and try to substantiate your conjecture.

SOLUTIONS TO DISCUSSION PROBLEMS AND ACTIVITIES:

I. In inductive reasoning, we reason to a general conclusion via the observations of specific cases. The conclusions obtained via inductive reasoning are only probable but not absolutely certain. In contrast, deductive reasoning is the method of reasoning to a specific conclusion through the use of general observations. The conclusions obtained through the use of deductive reasoning are certain. In the following number puzzle, we employ the five specific numbers 5, 23, 12, 10, and 85 to illustrate inductive reasoning and then employ algebra to furnish a deductive proof. The puzzle and the solutions are provided below:

Pick any Number.	5	23	12	10	85	n
Add 221 to the given selected number.	226	244	233	231	306	$n+221$
Multiply the sum by 2652.	599352	647088	617916	612612	811512	$2652n+586092$
Subtract 1326 from your product.	598026	645762	616590	611286	810186	$2652n+584776$
Divide your difference by 663.	902	974	930	922	1222	$4n+882$
Subtract 870 from your quotient.	32	104	60	52	352	$4n+12$
Divide your difference by 4.	8	26	15	13	88	$n+3$
Subtract the original number from your quotient.	3	3	3	3	3	3

The answer we obtain is always 3. We next deploy the calculator to show the inductive cases in **FIGURES 1-10** and the deductive case in **FIGURES 11-12:**

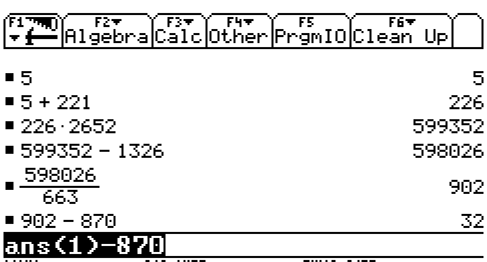


FIGURE 1

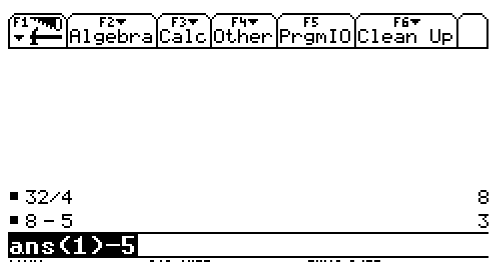


FIGURE 2

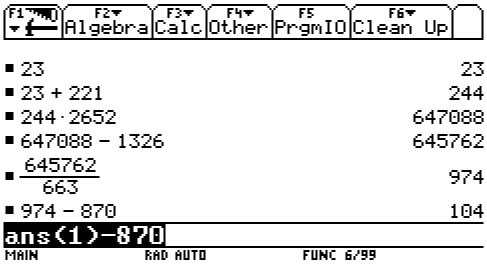


FIGURE 3

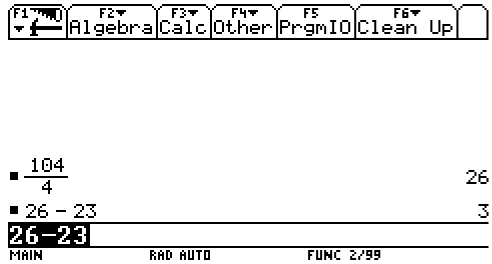


FIGURE 4

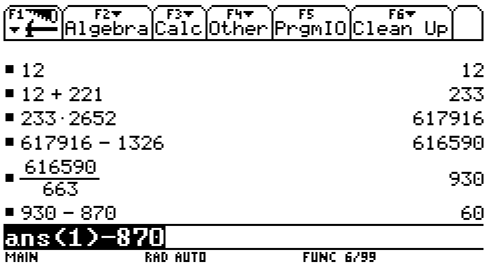


FIGURE 5

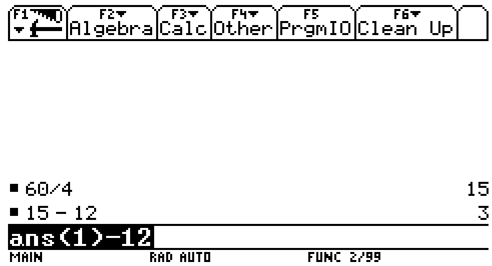


FIGURE 6

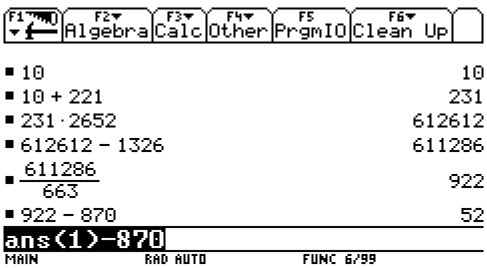


FIGURE 7

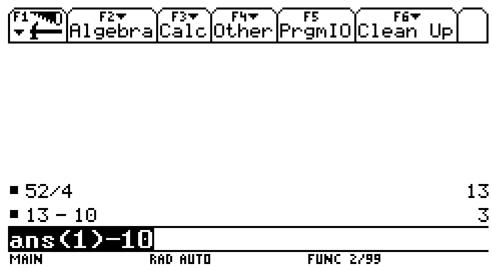


FIGURE 8

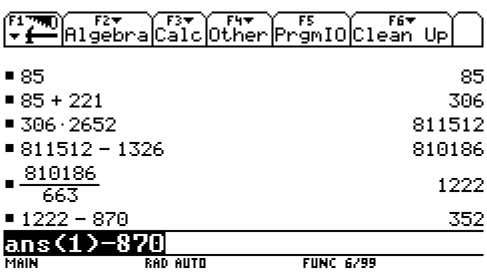


FIGURE 9

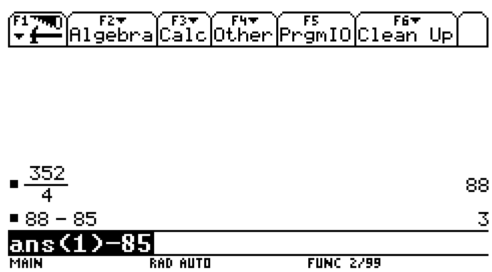


FIGURE 10

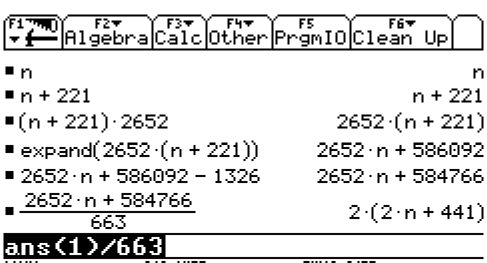


FIGURE 11

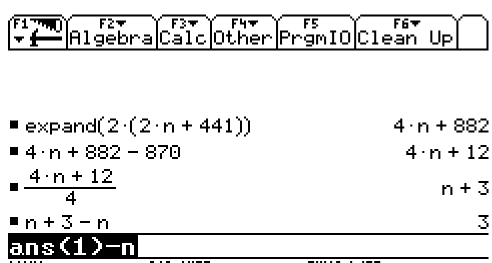


FIGURE 12

II. If we add successive unit fractions with even denominators, starting with $\frac{1}{2}$, we obtain the following for fifteen iterations in **FIGURES 13-15:**

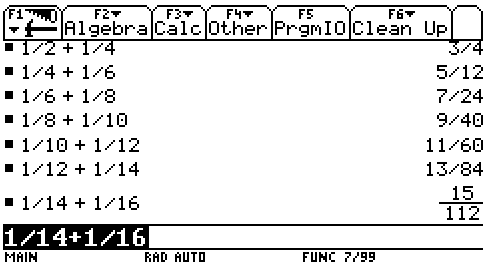


FIGURE 13

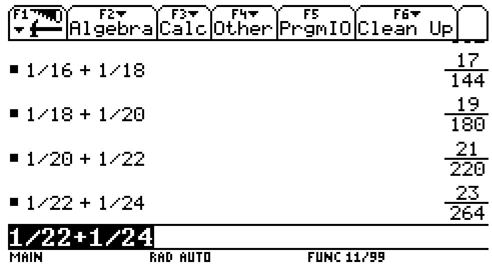


FIGURE 14

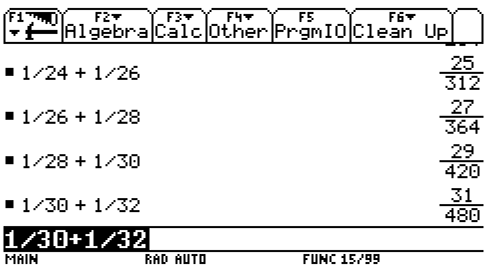


FIGURE 15

Similarly if we add successive unit fractions with odd denominators starting with 1/3, we obtain the following for fifteen iterations in **FIGURES 16-18:**

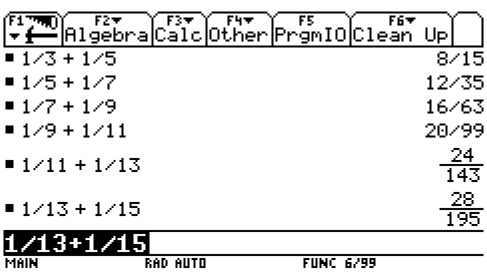


FIGURE 16

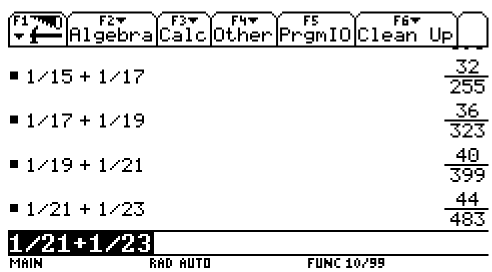


FIGURE 17

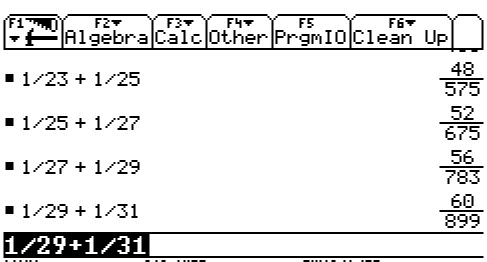


FIGURE 18

Consider the numerators and denominators of the sums obtained in **FIGURES 13-18.** We find that all form the legs of Primitive Pythagorean Triangles. In **FIGURES 19-28,** we show that the sum of the squares of the numerators and denominators of each of these fractions is a perfect square and hence a Pythagorean Triple is formed. Moreover, since $(a,b,c)=1$ in the sense that there are no common integer factors other than 1 among the components, the triples are classified as Primitive Pythagorean triples.

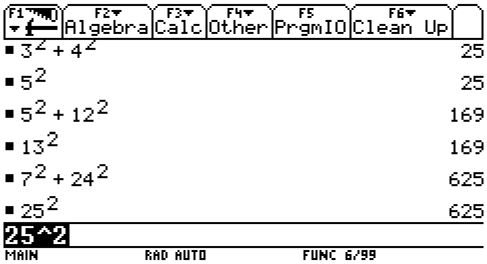


FIGURE 19

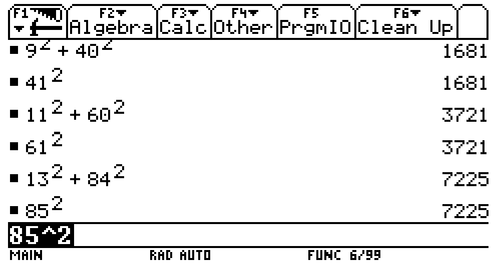


FIGURE 20

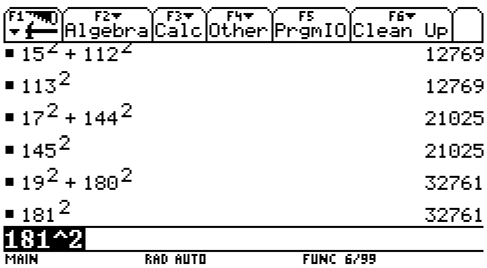


FIGURE 21

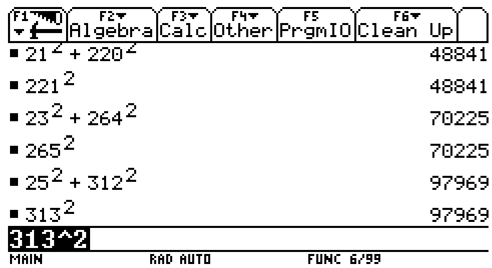


FIGURE 22

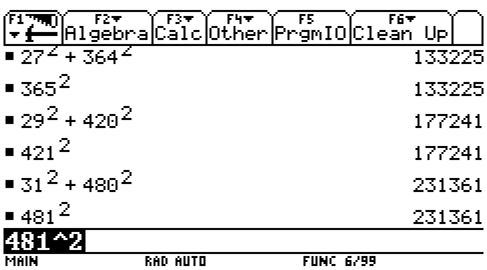


FIGURE 23

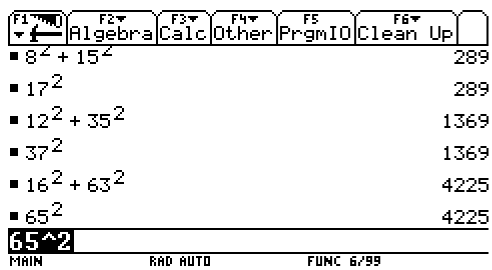


FIGURE 24

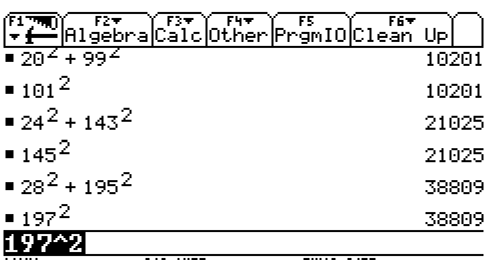


FIGURE 25

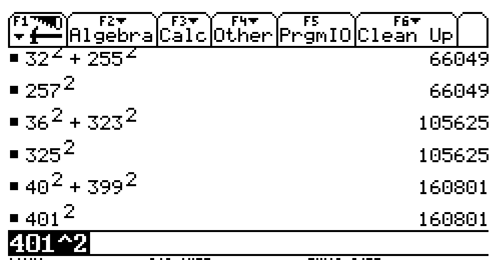


FIGURE 26

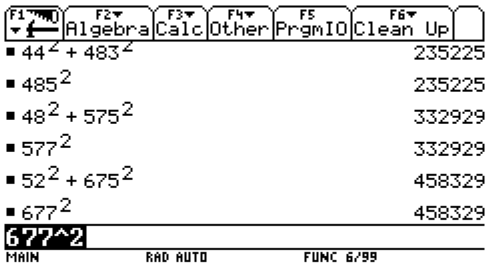


FIGURE 27

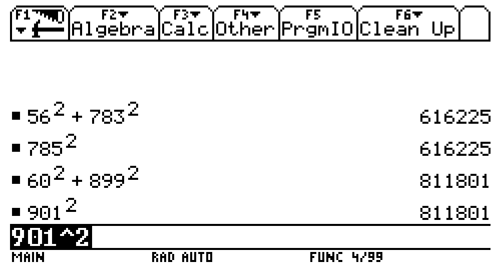


FIGURE 28

Are these results always true and are all Primitive Pythagorean Triples obtained in this manner? The answers are YES and NO respectively. The latter question can be resolved by noting that the PPT (77, 36, 85) is not generated by this process. See **FIGURE 29.** In addition, view **FIGURES 31-34** for a general proof considering the cases of the sum of consecutive even unit fractions and the sum of consecutive odd unit fractions separately with **FIGURE 30** displaying the relevant Algebra Menu F2 on the TI-89/VOYAGE 200.

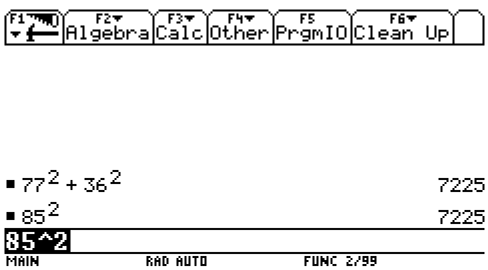


FIGURE 29

One can show that the following is true in general. If we consider that any even integer is of the form $2\cdot m$ for some integer m and any odd integer is of the form $2\cdot n+1$ for some integer n, then we have the following for the sum of two unit fractions with consecutive even denominators:

$\frac{1}{2\cdot m}+\frac{1}{2\cdot m+2}=\frac{2\cdot m+1}{2\cdot m^2+2\cdot m}$. Taking the sum of the squares of the numerator and denominator of this fraction, we note

$(2\cdot m+1)^2+\left(2\cdot m^2+2\cdot m\right)^2=4\cdot m^4+8\cdot m^3+8\cdot m^2+4\cdot m+1=\left(2\cdot m^2+2\cdot m+1\right)^2$. Hence one obtains the Primitive Pythagorean triple $\left(2\cdot m+1,\ 2\cdot m^2+2\cdot m,\ 2\cdot m^2+2\cdot m+1\right)$.

For the sum of two unit fractions with consecutive odd denominators, we observe the following:

$\frac{1}{2\cdot n+1}+\frac{1}{2\cdot n+3}=\frac{4\cdot n+4}{4\cdot n^2+8\cdot n+3}$. Taking the sum of the squares of the numerator and denominator of this fraction, we note that

$(4\cdot n+4)^2+(4\cdot n^2+8\cdot n+3)^2=16\cdot n^4+64\cdot n^3+104\cdot n^2+80\cdot n+25=(4\cdot n^2+8\cdot n+5)^2$. Thus one obtains the Primitive Pythagorean Triple $(4\cdot n+4,\ 4\cdot n^2+8\cdot n+3,\ 4\cdot n^2+8\cdot n+5)$.

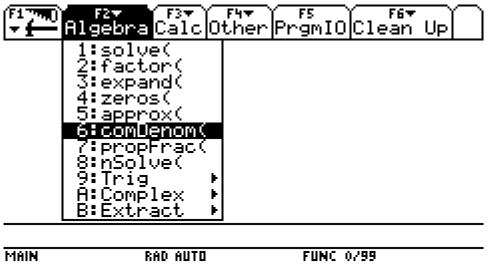


FIGURE 30

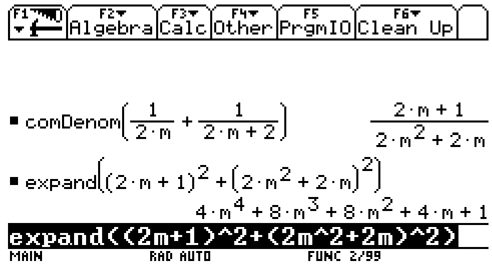


FIGURE 31

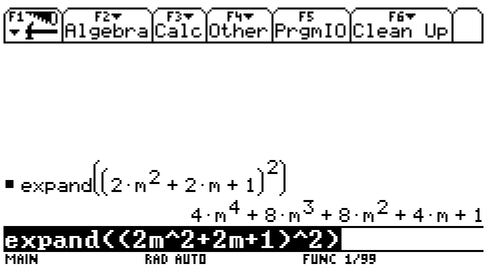


FIGURE 32

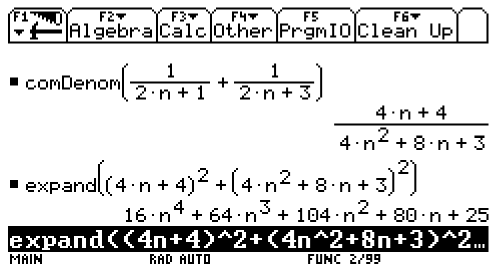


FIGURE 33

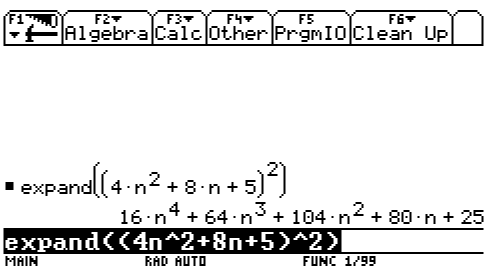
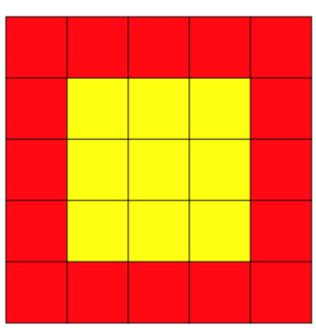
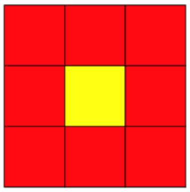


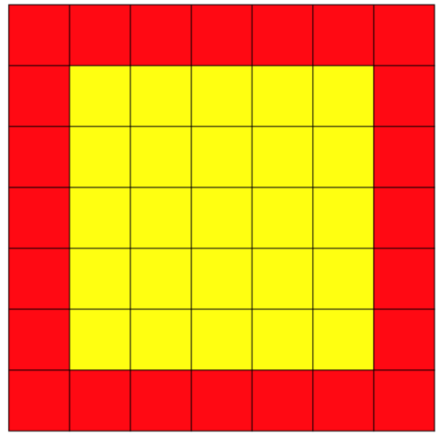
FIGURE 34

III. (a). Observe that the first figure consists of nine squares, eight of which are shaded red along the border and one that is shaded yellow in the center. The second figure consists of twenty-five squares of which sixteen are shaded red along the border and nine that are shaded yellow. The third figure is composed of forty-nine squares of which twenty-four are shaded red along the border and twenty-five that are shaded yellow. The next two figures would respectively eighty-one squares, thirty-two which are shaded red along the border and forty-nine shaded yellow followed by one hundred twenty-one squares, forty of which are red along the border and eighty-one that are shaded yellow. See the figures below:





1^{st} ____________ 2^{nd}



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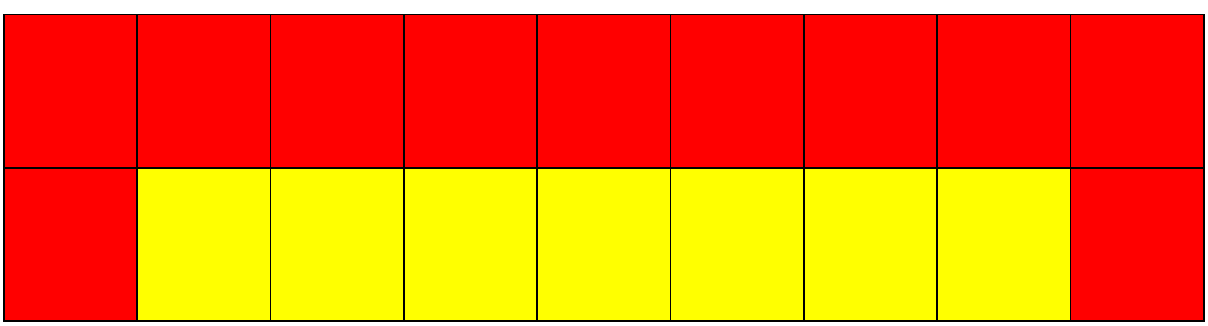
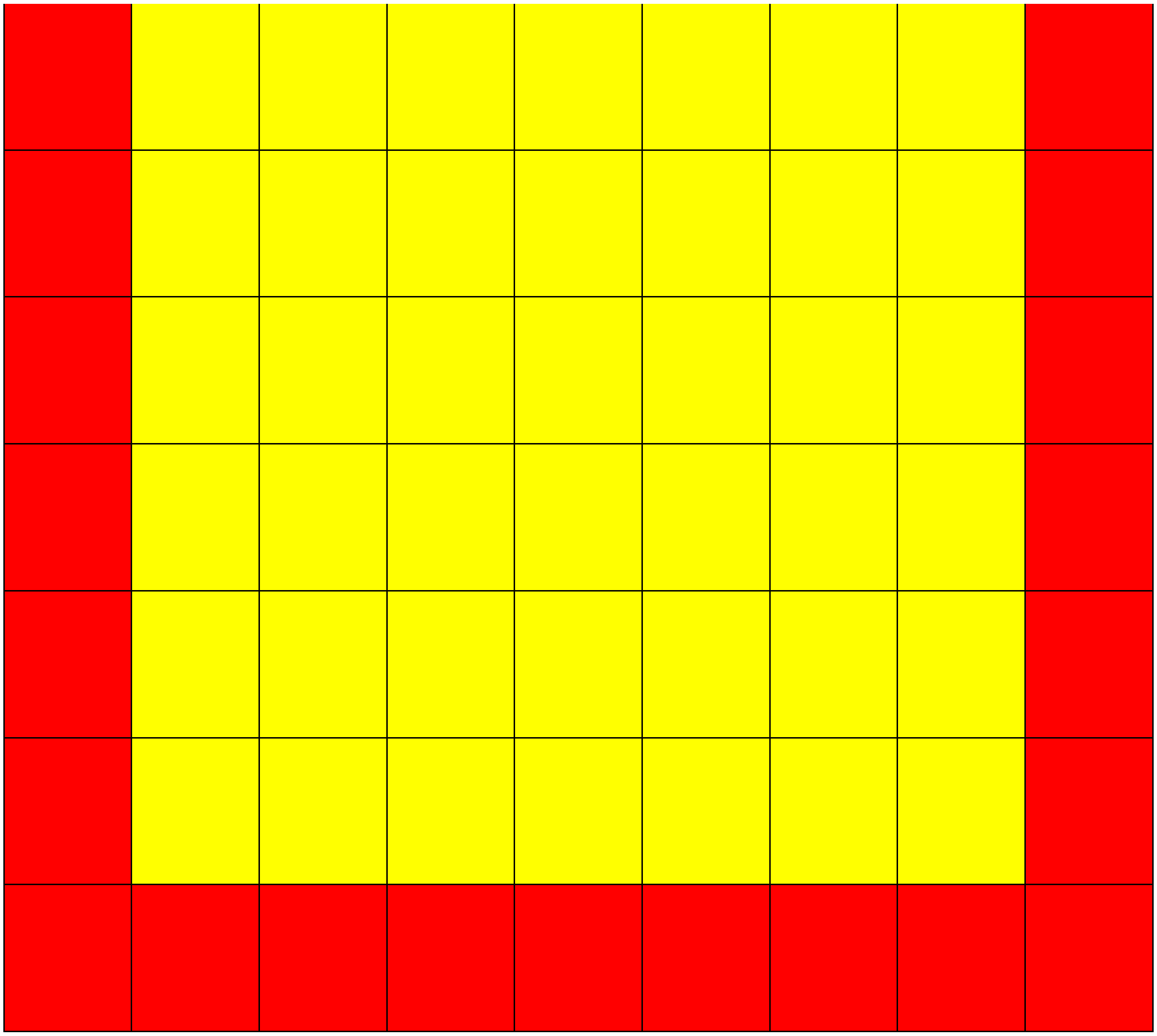
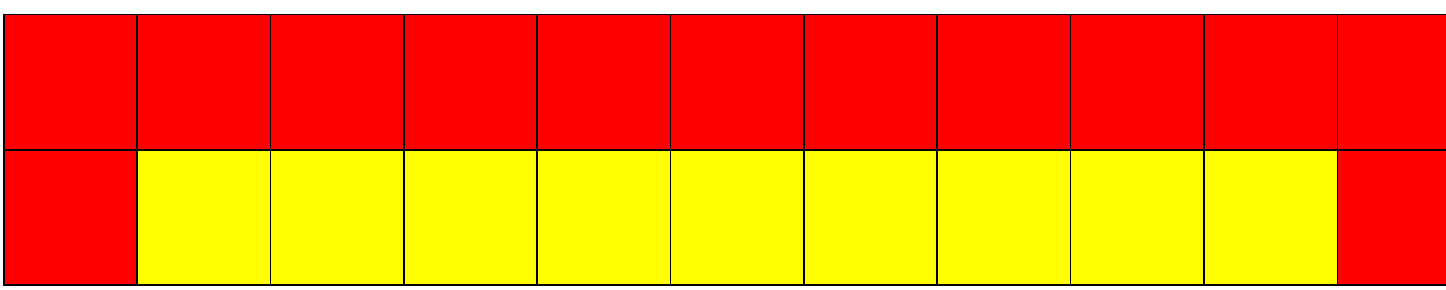
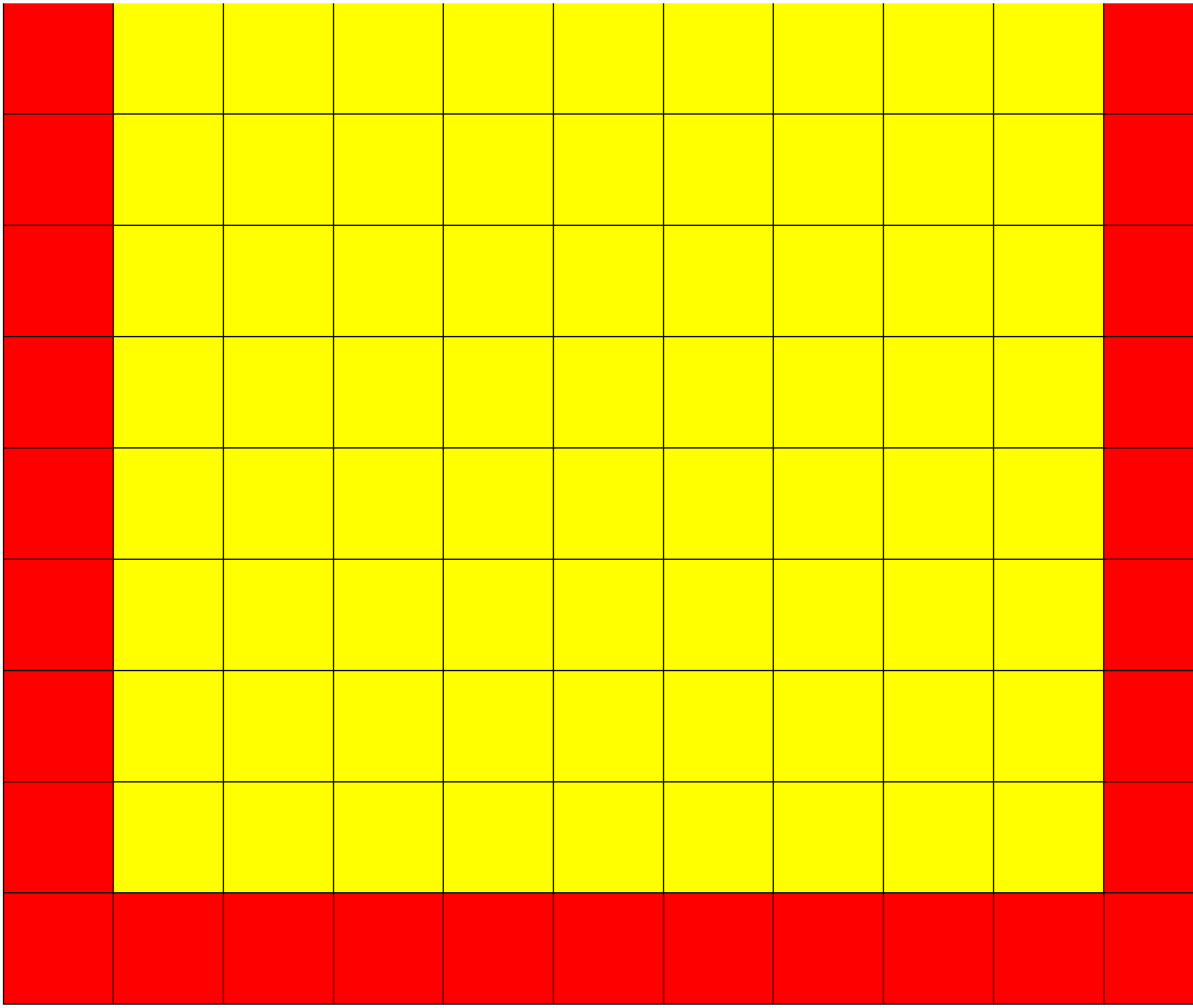
3^{rd} ____________





4th





5th

(b). Following the above pattern, the 100th figure would consist of forty thousand, four hundred one squares of which eight hundred squares are red on the border and thirty-nine thousand six hundred one are yellow. Let us create a table and show the first ten iterations with the number of squares of each color as well as the total number of squares to detect a pattern:

Iteration	Number of Red Squares	Number of Yellow Squares	Total Number of Squares
1st	8	1	9

2^{nd}	16	9	25
3^{rd}	24	25	49
4^{th}	32	49	81
5^{th}	40	81	121
6^{th}	48	121	169
7^{th}	56	169	225
8^{th}	64	225	289
9^{th}	72	289	361
10^{th}	80	361	441
---	---	---	---
100^{th}	800	39601	40401
---	---	---	---
$n-th$	$8\cdot n$	$(2\cdot n-1)^2$	$(2\cdot n+1)^2$

Observe that the number of red squares coincides with eight times the iteration number while the number of yellow squares is the square of one less than twice the iteration number and the total number of squares is one more than twice the iteration number. Note that

$8\cdot n+(2\cdot n-1)^2=8\cdot n+4\cdot n^2-4\cdot n+1=4\cdot n^2+4\cdot n+1=(2\cdot n+1)^2$.

IV. Let us obtain empirical evidence utilizing a graphing calculator. The unit's digits of perfect squares can be seen in **FIGURES 35-38:**

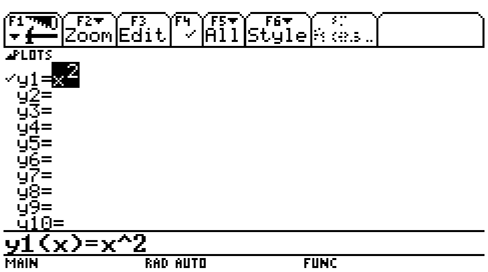


FIGURE 35

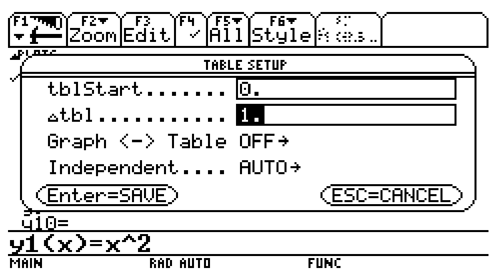


FIGURE 36

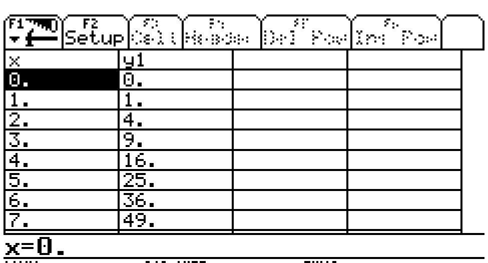


FIGURE 37

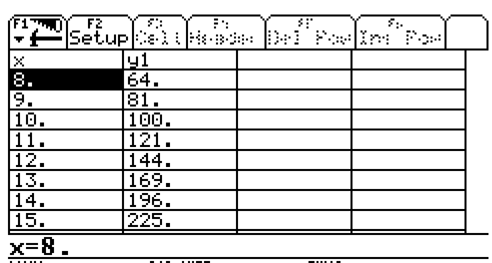


FIGURE 38

Hence we see that the unit's digits form a cycle 0, 1, 4, 9, 6, 5, 6, 9, 4, 1 of period 10 before repeating. Note that since 2, 3, 7 and 8 do not appear, these digits cannot be the last digits of integers that are perfect squares. Note the palindromic nature of the sequence 0, 1, 4, 9, 6, 5, 6, 9, 4, 1, 0 in the sense that the sequence reads the same both forwards and backwards.

For the last two digits of integers to be candidates for perfect squares, our investigation must be pursued more deeply. See **FIGURES 39-45:**

x	y1
0.	0.
1.	1.
2.	4.
3.	9.
4.	16.
5.	25.
6.	36.
7.	49.

x=0.

FIGURE 39

x	y1
8.	64.
9.	81.
10.	100.
11.	121.
12.	144.
13.	169.
14.	196.
15.	225.

x=8.

FIGURE 40

x	y1
16.	256.
17.	289.
18.	324.
19.	361.
20.	400.
21.	441.
22.	484.
23.	529.

x=16.

FIGURE 41

x	y1
24.	576.
25.	625.
26.	676.
27.	729.
28.	784.
29.	841.
30.	900.
31.	961.

x=24.

FIGURE 42

x	y1
32.	1024.
33.	1089.
34.	1156.
35.	1225.
36.	1296.
37.	1369.
38.	1444.
39.	1521.

x=32.

FIGURE 43

x	y1
40.	1600.
41.	1681.
42.	1764.
43.	1849.
44.	1936.
45.	2025.
46.	2116.
47.	2209.

x=40.

FIGURE 44

x	y1
48.	2304.
49.	2401.
50.	2500.
51.	2601.
52.	2704.
53.	2809.
54.	2916.
55.	3025.

x=50.

FIGURE 45

Hence the last two digits of integers which are perfect squares forms a cycle of length fifty as follows: 00, 01, 04, 09, 16, 25, 36, 49, 64, 81, 00, 21, 44, 69, 96, 25, 56, 89, 24, 61, 00, 41, 84, 29, 76, 25, 76, 29, 84, 41, 00, 61, 24, 89, 56, 25, 96, 69, 44, 21, 00, 81, 64, 49, 36, 25, 16, 09, 04, 01,… Observe that after the 25^{th}, the sequence reads the same backwards. Based on the above, it is evident that 742203 cannot be a perfect square candidate while 113569, 90687529, 23853456 and 9354676 are all possible candidates for perfect squares. To determine if these integers are actually perfect squares, we take their square roots displayed in **FIGURE 46:**

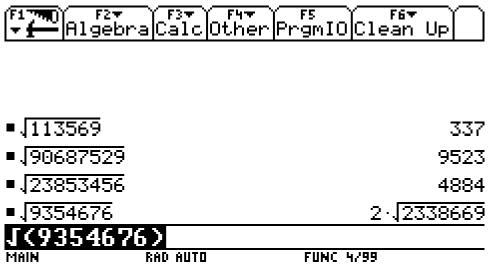


FIGURE 46

An analysis of **FIGURE 46** indicates that 113569, 90687529 and 23853456 are perfect squares while 9354676 is not a perfect square.

V. We next consider the sequence 9, 98, 987, 9876, 98765, 987654,… and determine which integers in the sequence are divisible by each of the first eleven counting integers. We seek a pattern and finally determine if there are any primes in the sequence.

A **MATHEMATICA** investigation was launched as well as one with a CAS calculator. See **FIGURES 47-52** for the calculator investigation:

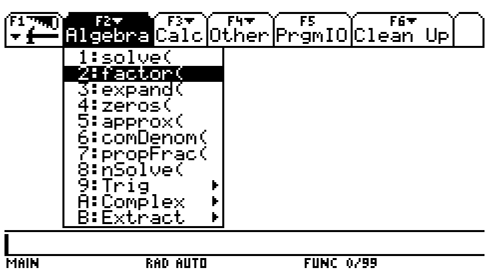


FIGURE 47

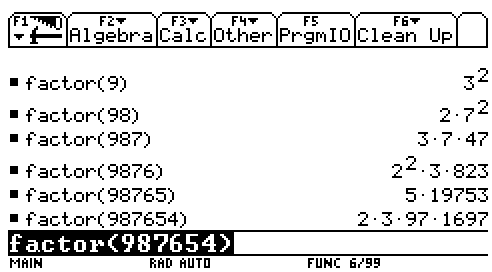


FIGURE 48

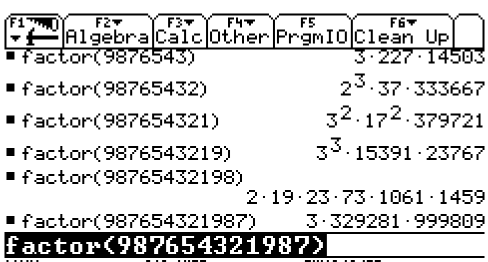


FIGURE 49

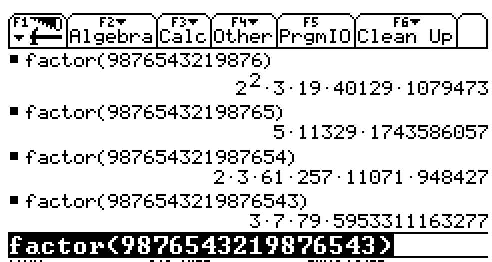


FIGURE 50

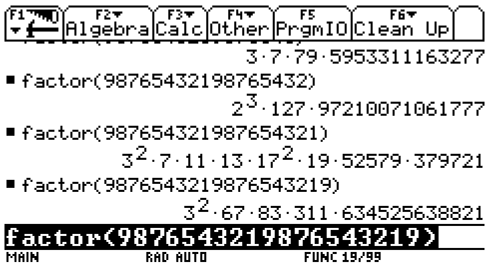


FIGURE 51

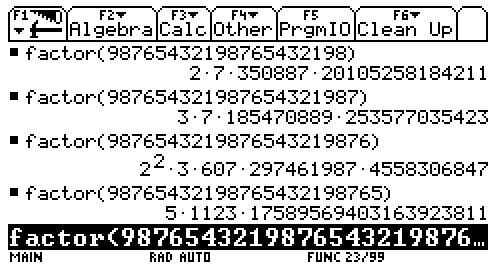


FIGURE 52

There are no primes in the sequence 9, 98, 987, 9876, 98765, 987654, 9876543, 98765432, 987654321, 9876543219, 98765432198,…. Clearly the second, fourth, sixth, eighth, and eleventh terminate in an even digit and are divisible by two. Every third integer is divisible by three beginning with the third iteration (the sum of the digits of these integers is divisible by 3). Since the initial integer 9 is divisible by 3 and we are adding three consecutive integers (whose sum is certainly divisible by 3), every integer of the form $3 \cdot k + 1,\ k \in W,$ (W is the set of whole numbers) is divisible by 3. Hence the first, fourth, seventh, tenth integers etc. are divisible by 3. In order for an integer to be divisible by 4, the integer consisting of the last two digits must be divisible by 4. Since the digits in this pattern are consecutive integers in descending order, the only cases to consider are when the last two digits of the integer are either 76 or 32. This occurs in the fourth iteration and every ninth integer thereafter. Those integers that are divisible by 6 occur in the sixth iteration and every ninth thereafter as well as in the sixth iteration and every ninth thereafter. These iterations are of the form $9 \cdot k - 5$ *or* $9 \cdot k - 3,\ k \in \mathbb{N}$. Hence our initial few that are divisible by 6 are the fourth, sixth, thirteenth, fifteenth, twenty-second and twenty-fourth. For divisibility of an integer by 8, examine the number consisting of the last three digits. Only an iteration that terminates in the consecutive digits 432 will work. This is true in the eighth iteration and every ninth one thereafter. For divisibility of an integer by 9, the sum of the digits must be divisible by 9. This is true for the ninth iteration (987654321) and every ninth one thereafter. That is, iterations of the form $9 \cdot k,\ k \in \mathbb{N}$. No member of the sequence is divisible by 10 since the last digit is never 0. In this pattern, observe that 9 is divisible by 3, 98 is even, 987 is divisible by 3, 9876 is even, 98765 is divisible by 5, 987654 is even, 9876543 is divisible by 3, 98765432 is even and 987654321 is divisible by 3. Appending the digit 9 to 987654321 renders the integer 9876543219 divisible by 3 and the pattern continues. Thus there are no primes in the sequence.

VI. The graphing calculator can serve as an aid in the investigation. We consider the output values for n^{10} *and* 2^n. For the initial cases, the power function is consistently larger than the exponential function contrary to expectations. The roles eventually are reversed. See **FIGURES 53-72:**

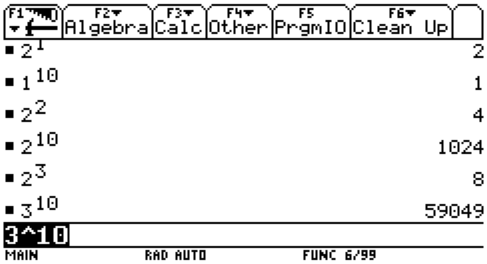


FIGURE 53

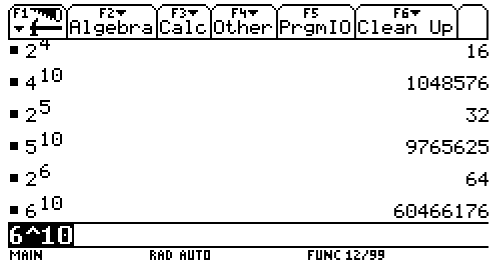


FIGURE 54

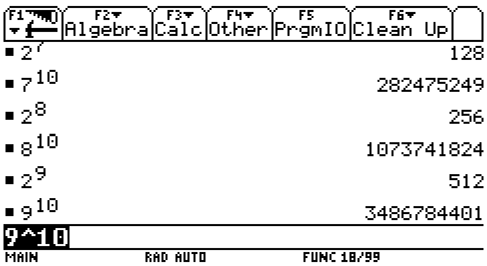


FIGURE 55

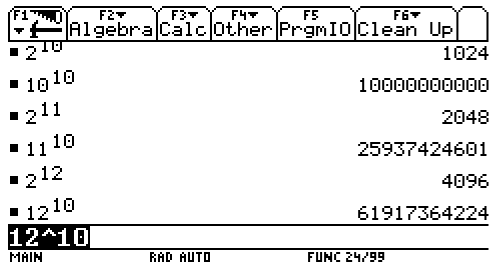


FIGURE 56

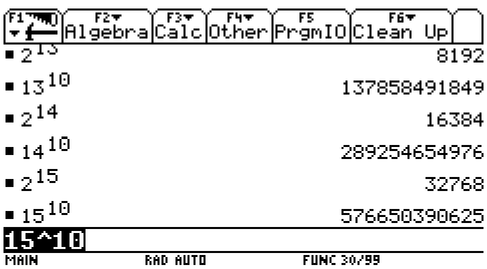


FIGURE 57

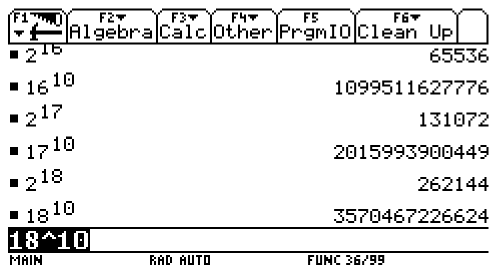


FIGURE 58

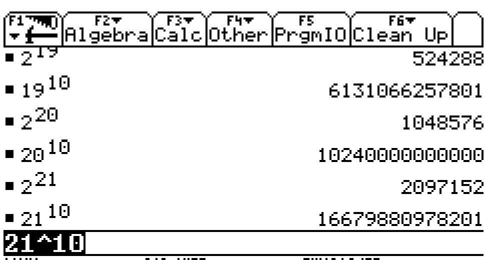


FIGURE 59

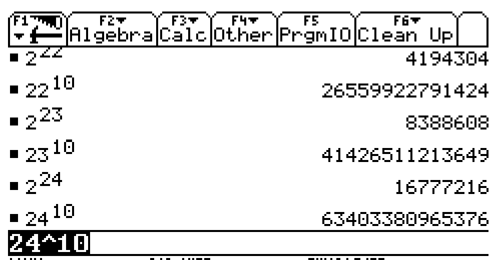


FIGURE 60

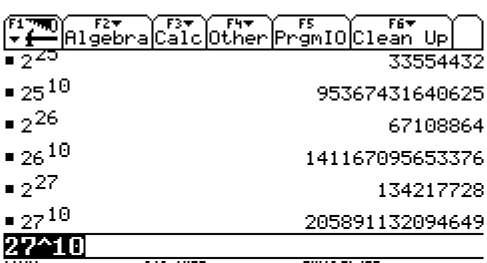


FIGURE 61

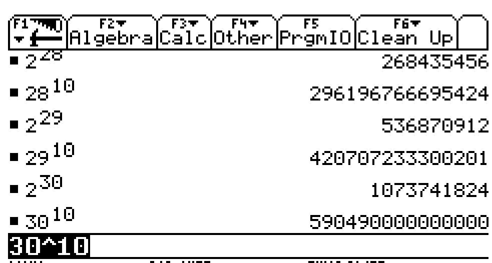


FIGURE 62

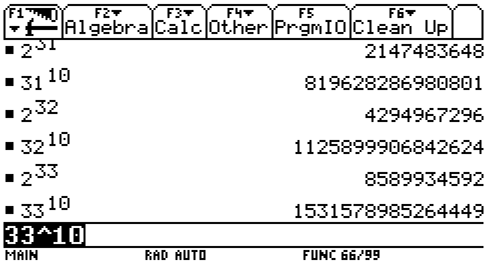


FIGURE 63

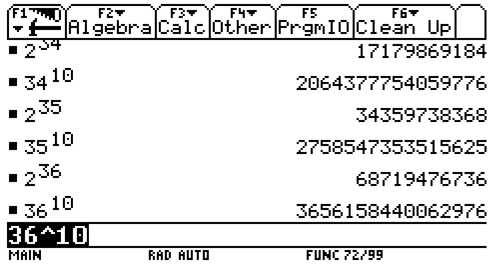


FIGURE 64

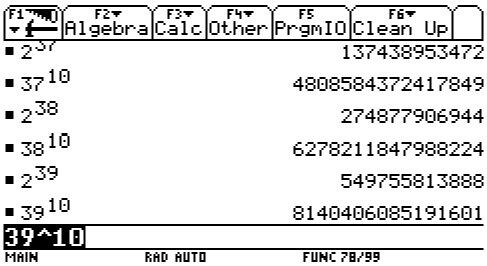


FIGURE 65

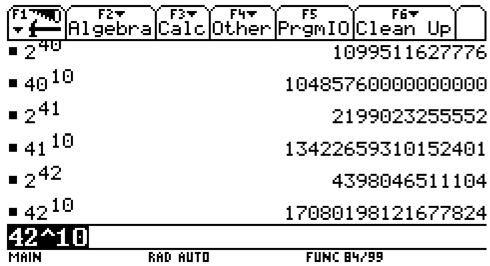


FIGURE 66

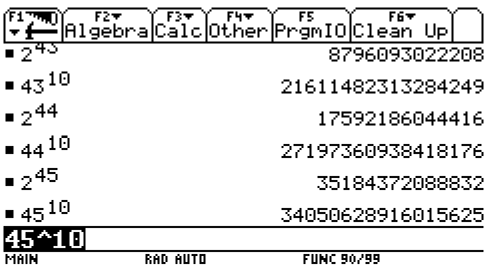


FIGURE 67

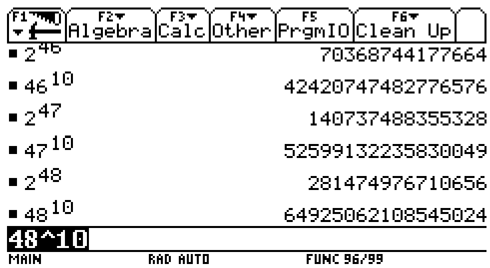


FIGURE 68

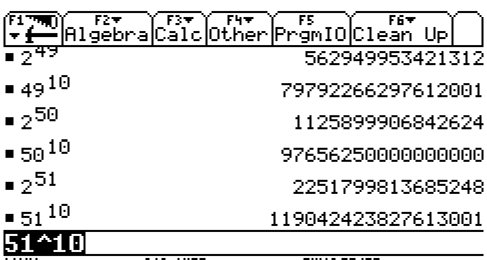


FIGURE 69

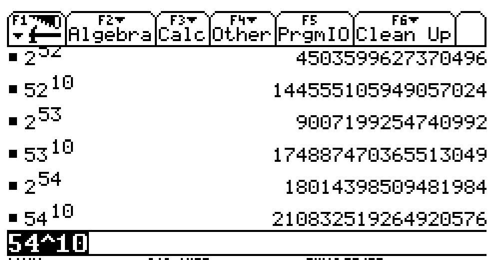


FIGURE 70

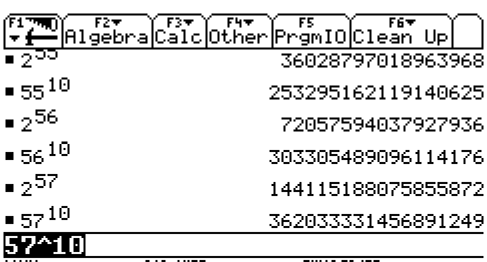


FIGURE 71

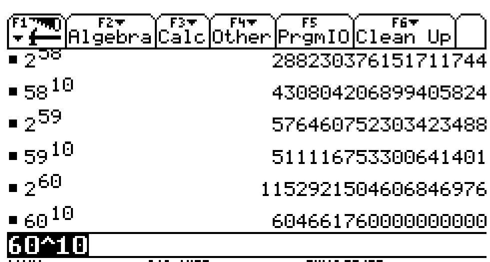


FIGURE 72

It is thus evident that $n = 59$ the initial integer such that $2^n > n^{10}$. Also see **FIGURES 73-82:**

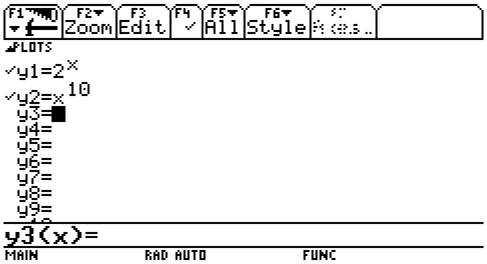


FIGURE 73

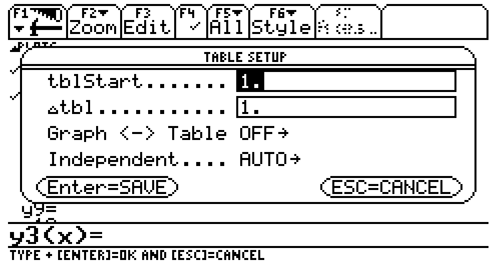
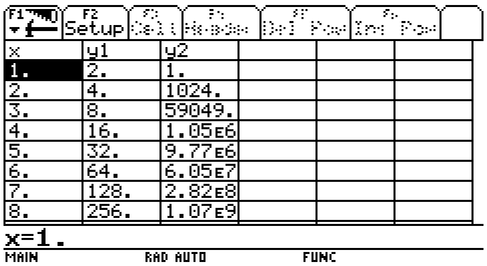
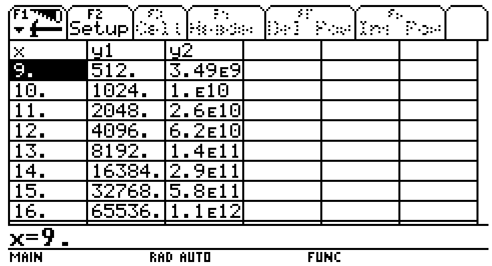


FIGURE 74



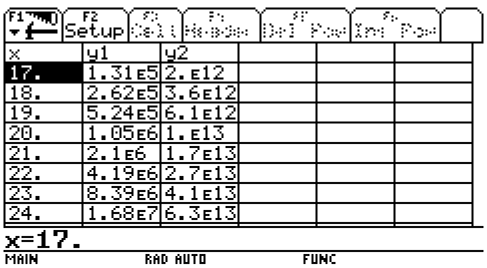
x	y1	y2
1.	2.	1.
2.	4.	1024.
3.	8.	59049.
4.	16.	1.05E6
5.	32.	9.77E6
6.	64.	6.05E7
7.	128.	2.82E8
8.	256.	1.07E9

FIGURE 75



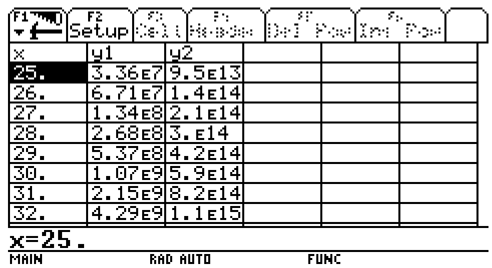
x	y1	y2
9.	512.	3.49E9
10.	1024.	1.E10
11.	2048.	2.6E10
12.	4096.	6.2E10
13.	8192.	1.4E11
14.	16384.	2.9E11
15.	32768.	5.8E11
16.	65536.	1.1E12

FIGURE 76



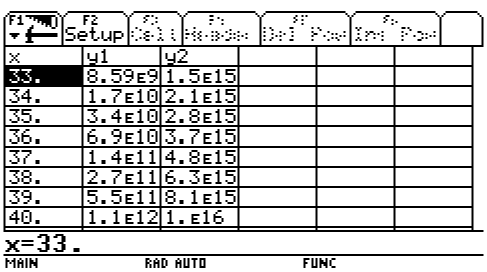
x	y1	y2
17.	1.31E5	2.E12
18.	2.62E5	3.6E12
19.	5.24E5	6.1E12
20.	1.05E6	1.E13
21.	2.1E6	1.7E13
22.	4.19E6	2.7E13
23.	8.39E6	4.1E13
24.	1.68E7	6.3E13

FIGURE 77



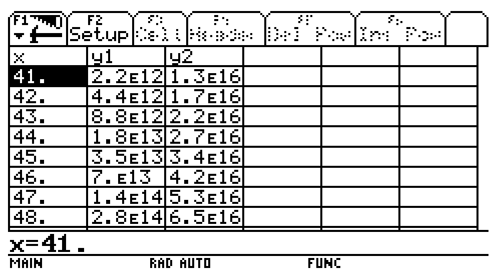
x	y1	y2
25.	3.36E7	9.5E13
26.	6.71E7	1.4E14
27.	1.34E8	2.1E14
28.	2.68E8	3.E14
29.	5.37E8	4.2E14
30.	1.07E9	5.9E14
31.	2.15E9	8.2E14
32.	4.29E9	1.1E15

FIGURE 78



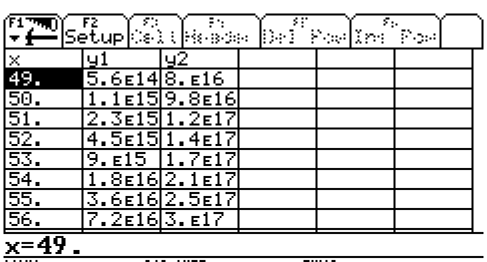
x	y1	y2
33.	8.59E9	1.5E15
34.	1.7E10	2.1E15
35.	3.4E10	2.8E15
36.	6.9E10	3.7E15
37.	1.4E11	4.8E15
38.	2.7E11	6.3E15
39.	5.5E11	8.1E15
40.	1.1E12	1.E16

FIGURE 79



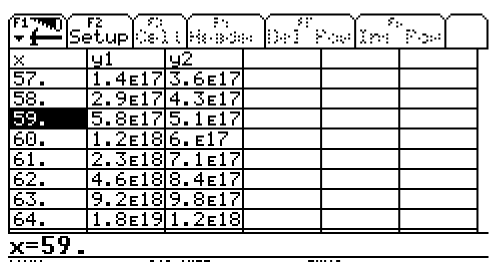
x	y1	y2
41.	2.2E12	1.3E16
42.	4.4E12	1.7E16
43.	8.8E12	2.2E16
44.	1.8E13	2.7E16
45.	3.5E13	3.4E16
46.	7.E13	4.2E16
47.	1.4E14	5.3E16
48.	2.8E14	6.5E16

FIGURE 80



x	y1	y2
49.	5.6E14	8.E16
50.	1.1E15	9.8E16
51.	2.3E15	1.2E17
52.	4.5E15	1.4E17
53.	9.E15	1.7E17
54.	1.8E16	2.1E17
55.	3.6E16	2.5E17
56.	7.2E16	3.E17

FIGURE 81



x	y1	y2
57.	1.4E17	3.6E17
58.	2.9E17	4.3E17
59.	5.8E17	5.1E17
60.	1.2E18	6.E17
61.	2.3E18	7.1E17
62.	4.6E18	8.4E17
63.	9.2E18	9.8E17
64.	1.8E19	1.2E18

FIGURE 82

VII. Consider the following three sequences consisting of four consecutive Fibonacci numbers $\{3,5,8,13\}$, $\{8,13,21,34\}$ *and* $\{13,21,34,55\}$. See **FIGURES 83-85:**

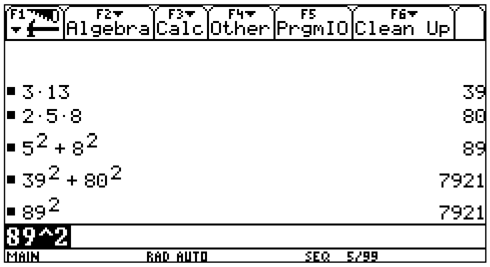


FIGURE 83

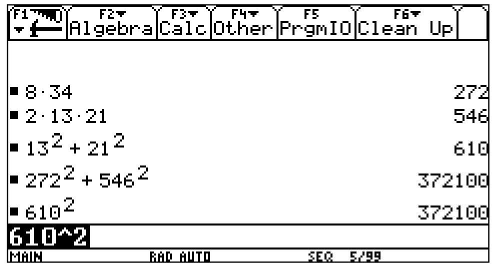


FIGURE 84

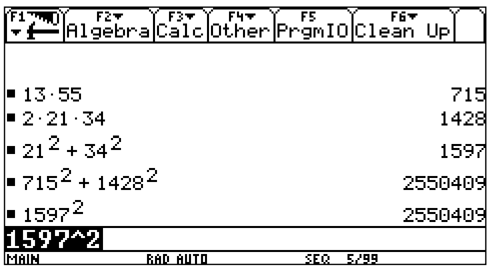


FIGURE 85

Observe that the respective Pythagorean Triples (39, 80, 89), (272, 546, 610) and (715, 1428, 1597) are formed. The first and third of these Pythagorean triples are primitive. In contrast, the Pythagorean Triple (272, 546, 610) is not primitive; for 2 is a common factor among each of the components. The associated primitive Pythagorean Triple is (136, 273, 305). Note that the hypotenuses of each of the right triangles formed are the Fibonacci numbers 89, 610 and 1597. Based on the observations in the three examples, one might suspect that a Pythagorean triple is always formed. This is indeed the case. We justify our conjecture with the VOYAGE 200: Suppose $\{x, y, x+y, x+2\cdot y\}$ represent any four consecutive terms of the Fibonacci (or Fibonacci-like sequence). We view our inputs and outputs in **FIGURE 87** using the expand command (See **FIGURE 86)** from the Algebra menu on the HOME SCREEN:

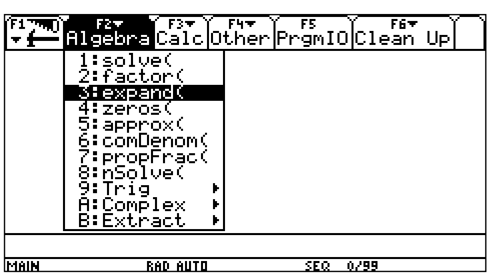


FIGURE 86

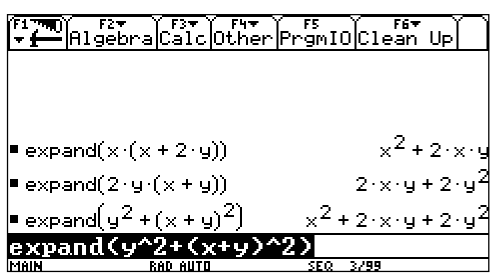


FIGURE 87

To show that $\left(x^2+2\cdot x\cdot y, 2\cdot x\cdot y+2\cdot y^2, x^2+2\cdot x\cdot y+2\cdot y^2\right)$ forms a Pythagorean Triple, see **FIGURES 88-90** for our inputs and outputs below:

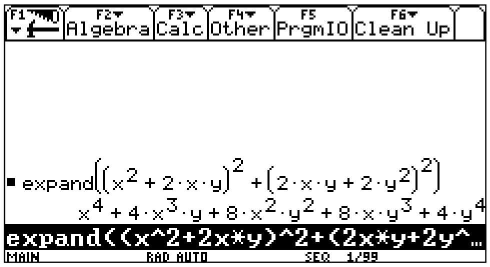


FIGURE 88

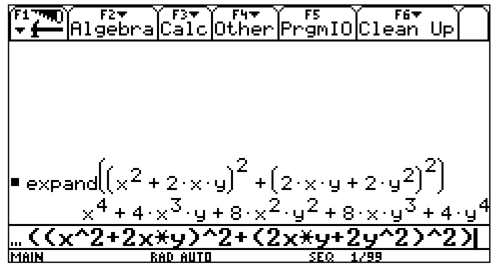


FIGURE 89

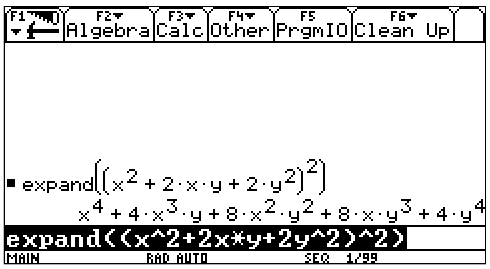


FIGURE 90

THANK YOU FOR YOUR PARTICIPATION AT THIS WORKSHOP DURING THE 7TH ANNUAL SPECIAL EDUCATION AND MATHEMATICS CONFERENCE AT ST. PETERS UNIVERSITY IN JERSEY CITY, NJ!