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http://www.amtnj.org
Editorial Fall, 2012

Tom Walsh, Editor-in-chief,
New Jersey Mathematics Teacher

It certainly has been a wonderful start to school this year. Mathematics continues to be a vital part of students’ education, and the new Common Core State Standards are being implemented by more and more New Jersey districts. I hope you will take advantage of the great professional development opportunities offered by AMTNJ for learning about the Common Core State Standards. We have some truly dedicated instructors in this field, and they are available to come to your school. Find them online at www.amtnj.org

I have been introduced to a wonderful organization that started up just very recently. The Museum of Mathematics (called MoMath) was started in 2008 with a capital campaign of over 20 million, and they will be opening their museum to the public in the Flatiron District of Manhattan in December of this year. In early 2009, the MoMath team put together a Math Midway, part of the World Science Fair, held in Washington Square Park. They developed some wonderful exhibits, like a tricycle with square wheels that rides on a large circular track made of semi-circular humps that exactly fit the square wheels. The tricycle rides very smoothly! Another exhibit was the Number Line Tightrope (numbering to 100), with over ten families of numbers (ex: primes, odds, evens) with colorful tags for people to follow the patterns.

The Math Midway came to the Liberty Science Center from October – December, 2011. I had a great time looking at all the exhibits, and I learned a lot from the 21 exciting activities. It’s nice to see a dedicated group of mathematicians making math topics fun and engaging. Go to MoMath.org for more information, and consider joining the group. They can always use people who love math.

This issue has several great articles. The first one is a look at Pythagorean Triples, written by Fred Chichester, an engineer by training, and a man who has been studying math his whole life. He’s taught math topics in a few schools around northern NJ as well, and he’s the husband of Pat Kenschaft who has contributed a few articles to this Journal.

Our second article is from a recently retired professor from The Ohio State University. Edward Laughbaum discusses how learning new things (and providing connections between the new topic and previous knowledge) can enhance the brain’s long-term memory. He discusses how important the timing is for making the connections between the concept and a student’s memory, and he suggests some ideas for teaching students in a math class.

In our third article, Farshid Safir and Erin Suozzo discuss an informal introduction to limits. They give a few nice ways to understand the rather difficult concept of limits, which can be a serious problem for some students. They offer ideas for
teaching this concept in high school, where it is seldom introduced, but where it should be.

The fourth article discusses solving ratio problems. Rowan University’s Karen Heinz points out how ratio and proportion problems can be represented in three other ways. These other ways can help you reach a great number of your students who might not otherwise grasp the concept through proportions alone.

As always, we have announcements about our upcoming conferences, and solicitations to join and/or volunteer and/or contribute to this publication. Please consider helping your association. Details are in the announcement at the end of the journal. If you have something on your mind, you can contact me or any member of the staff at the email addresses in this issue.
Squares and Pythagorean Triples
Fred Chichester

Abstract:

Six equations are developed from which the squares of all of the integers between 10 and 100 can be calculated mentally. The results of these calculations are then used to identify sixteen primitive Pythagorean triples, each of which represents the lengths of the sides of the corresponding Pythagorean triangle.

Squaring numbers mentally is useful in generating Pythagorean triples, sets of three positive integers related in such a way that the sum of the squares of the two smaller ones in each triple equals the square of the largest one. The conventional notation for these integers, representing the lengths of the sides of a right triangle in order of increasing magnitude is: (a, b, c) where the integers are related by the following equation:

\[ a^2 + b^2 = c^2, \]  

which is equivalent to: \[ a^2 = c^2 - b^2. \]  

The Pythagorean triples for which a, b and c have no common factors are called primitive Pythagorean triples. They are considered to be fundamental because many other Pythagorean triples can be constructed from them of the form, (ma, mb, mc), where m is an integer other than one.

We begin by listing \(n^2\) for \(n = 1, 2, \ldots 9\): \(1^2 = 1, \ 2^2 = 4, \ 3^2 = 9, \ 4^2 = 16, \ 5^2 = 25, \ 6^2 = 36, \ 7^2 = 49, \ 8^2 = 64, \ 9^2 = 81.\)

For 2-digit numbers ending with zero: \((10m)^2 = m^2(100)\) for \(m = 1, 2, \ldots 10.\) \(10^2 = 1^2(100) = 100\) using equation (2) with \(m = 1\)

For 2-digit numbers ending in 5: 
\[(10m+5)^2 = 100^2 + 2(50)m + 5^2 = m(m+1)100 + 25\] for \(m = 1, 2, \ldots 9\)

For \(n = 11, 12, \ldots 9: \)
\(n = 10 + m: \ (10+m)^2 = 10^2 + 2(10)m + m^2 = 100 + 20m + m^2\) for \(m = 1, 2, \ldots 9\)
\((10 -m)^2 = 10^2 - 2(10)m + m^2 = 100 - 20m + m^2\) for \(m = 1, 2, \ldots 9\)

Subtracting the previous line from the one just above it, we get 
\((10+m)^2 = (10-m)^2 + 40m\) for \(m = n-10 = 1, 2, \ldots 9\)

\(11^2 = 9^2 + 40(1) = 81 + 40 = 121\)
12^2 = 8^2 + 40(2) = 64 + 80 = 144
13^2 = 7^2 + 40(3) = 49 + 120 = 169

15^2 = 1(2)100 + 25 = 200 + 25 = 225       using equation (3) with m = 1

16^2 = 4^2 + 40(6) = 16 + 240 = 256
17^2 = 3^2 + 40(7) = 9 + 280 = 289

By using the second form of equation (1) one can identify three primitive Pythagorean triples from the squares in the range of n = 3, 4, … 17 as follows.

5^2 – 4^2 = 25 – 16 = 9 = 3^2                          (3, 4, 5)
13^2 – 12^2 = 169 – 144 = 25 = 5^2                  (5, 12, 13)
17^2 – 15^2 = 289 – 225 = 64 = 8^2                (8, 15, 17)

20^2 = 2^2(100) = 400            using equation (2) for m = 2

For n = 21, 22, … 39:
We set n = 20 + m:  (20+m)^2 = 20^2 + 2(20)m + m^2 = 400 + 40m + m^2 for m = 1, 2, … 20
(20 -m)^2 = 20^2 - 2(20)m + m^2 = 400 - 40m + m^2 for m = 1, 2, … 20

Subtracting the previous line from the one just above it, we get
(20+m)^2 = (20-m)^2 + 80m       for m = 1, 2, … 20

Thus: (20+m)^2 = (20-m)^2 + 80m         for m = n - 20 = 1, 2, … 19       (5)

21^2 = 19^2 + 80(1) = 361 + 80 = 441
24^2 = 16^2 + 80(4) = 256 + 320 = 576
25^2 = 2(3)100 + 25 = 600 + 25 = 625       using equation (3) for m = 2
28^2 = 12^2 + 80(8) = 144 + 640 = 784
29^2 = 11^2 + 80(9) = 121 + 720 = 841
33^2 = 7^2 + 80(13) = 49 + 1,024 = 1,089
35^2 = 3(4)100 + 25 = 1,200 + 25 = 1,225       using equation (3) for m = 3
36^2 = 4^2 + 80(16) = 16 + 1,280 = 1,296
37^2 = 3^2 + 80(17) = 9 + 1,360 = 1,369
39^2 = 1^2 + 80(19) = 1 + 1,520 = 1,521
By using the second form of equation (1) one can identify **three additional** primitive Pythagorean triples from the squares in the range of \( n = 7, 8, \ldots 37 \) as follows.

\[
\begin{align*}
25^2 - 24^2 &= 625 - 576 = 49 = 7^2 & (7, 24, 25) \\
29^2 - 21^2 &= 841 - 441 = 400 = 20^2 & (20, 21, 29) \\
37^2 - 35^2 &= 1,369 - 1,225 = 144 = 12^2 & (12, 35, 37)
\end{align*}
\]

\[ 40^2 = 4^2(100) = 1,600 \quad \text{using equation (2) for } m = 4 \]

For \( n = 41, 42, \ldots 64 \): \((50+m)^2 = 2,500 + 100m + m^2 \) for \( m = n-50 = -9, -8, \ldots 14 \)  

\[
\begin{align*}
41^2 &= 2,500 - 9(100) + 9^2 = 1,600 + 81 = 1,681 \\
45^2 &= 4(5)100 + 25 = 2,000 + 25 = 2,025 \quad \text{using equation (3) for } m = 4 \\
48^2 &= 2,500 - 2(100) + 2^2 = 2,300 + 4 = 2,304 \\
53^2 &= 2,500 + 3(100) + 3^2 = 2,800 + 9 = 2,809 \\
55^2 &= 5(6)100 + 25 = 3,000 + 25 = 3,025 \quad \text{using equation (3) for } m = 5 \\
56^2 &= 2,500 + 6(100) + 6^2 = 3,100 + 36 = 3,136 \\
60^2 &= 6^2(100) = 3,600, \quad \text{using equation (2) for } m = 6 \\
61^2 &= 2,500 + 11(100) + 11^2 = 2,500 + 1,100 + 121 = 3,721 \\
63^2 &= 2,500 + 13(100) + 13^2 = 2,500 + 1,300 + 169 = 3,969 \\
65^2 &= 6(7)100 + 25 = 4,200 + 25 = 4,225 \quad \text{using equation (3) for } n = 6
\end{align*}
\]

By using the second form of equation (1) one can identify **five additional primitive Pythagorean triples** from the squares in the range of \( n = 9, 10, \ldots 65 \) as follows.

\[
\begin{align*}
41^2 - 40^2 &= 1,681 - 1,600 = 81 = 9^2 & (9, 40, 41) \\
53^2 - 45^2 &= 2,809 - 2,025 = 784 = 28^2 & (28, 45, 53) \\
61^2 - 60^2 &= 3,721 - 3,600 = 121 = 11^2 & (11, 60, 61) \\
65^2 - 63^2 &= 4,225 - 3,969 = 256 = 16^2 & (16, 63, 65) \\
65^2 - 56^2 &= 4,225 - 3,136 = 1,089 = 33^2 & (33, 56, 65)
\end{align*}
\]

For \( n = 66, 67, \ldots 99 \):

Set \( n = m + 50 \): \((50+m)^2 = 50^2 + 2(50)m + m^2 = 2,500 + 100m + m^2 \) for \( m = 16, 17, \ldots 49 \)

and \((50 - m)^2 = 50^2 - 2(50)m + m^2 = 2,500 - 100m + m^2 \) for \( m = 16, 17, \ldots 49 \)

Subtracting the previous line from the one just above it, we get
(50+m)^2 = (50-m)^2 + 200m for m = n-50 = 16, 17, … 49 \hspace{1cm} (7)

72^2 = 28^2 + 200(22) = 784 + 4,400 = 5,184
73^2 = 27^2 + 200(23) = 729 + 4,600 = 5,329
77^2 = 23^2 + 200(27) = 529 + 5,400 = 5,929
80^2 = 8^2(100) = 6,400, \hspace{1cm} \text{using equation (2) for m = 8}
84^2 = 16^2 + 200(34) = 256 + 6,800 = 7,056
85^2 = 8(9)100 + 25 = 7,200 + 25 = 7,225 \hspace{1cm} \text{using equation (3) for m = 8}
89^2 = 11^2 + 200(39) = 121 + 7,800 = 7,921
97^2 = 3^2 + 200(47) = 9 + 9,400 = 9,409

By using the second form of equation (1) one can identify four additional primitive Pythagorean triples from the squares in the range of n = 48, 49, … 97 as follows.

73^2 – 55^2 = 5,329 – 3,025 = 2,304 = 48^2 \hspace{1cm} (48, 55, 73)
85^2 – 77^2 = 7,225 – 5,929 = 1,296 = 36^2 \hspace{1cm} (36, 77, 85)
89^2 – 80^2 = 7,921 – 6,400 = 1,521 = 39^2 \hspace{1cm} (39, 80, 89)
97^2 – 72^2 = 9,409 – 5,184 = 4,225 = 65^2 \hspace{1cm} (65, 72, 97)

Conclusions

Knowing how to calculate squares of numbers mentally by using number patterns enables one to identify primitive Pythagorean triples representing the lengths of the sides of the corresponding right triangles.

In this article the concept of reflections about convenient pivot numbers such as 10, 20 and 50 to mentally calculate the square of a larger number from the known square of a smaller number that is the same distance from one of these pivot numbers is introduced and repeatedly applied. This technique facilitates easier identification of primitive Pythagorean triples.

An especially interesting result of this approach is the identification of the primitive Pythagorean triple, (20, 21, 29), because its two smaller numbers, which represent the legs of the corresponding Pythagorean triangle, identify a triangle that is near-isosceles.

Summary of Equations for Generating Squares of Numbers Between 10 and 100

For n = 10m: \hspace{1cm} (10m)^2 = m^2(100) \hspace{1cm} \text{for } m = 1, 2, \ldots, 10 \hspace{1cm} (2)
For \( n = 10m + 5 \): \( (10m + 5)^2 = 100m(m+1) + 25 \) for \( m = 1,2,\ldots 9 \) \hspace{1cm} (3)

For \( n = 11, 12,\ldots 19 \): \( (10+m)^2 = (10-m)^2 + 40m \) for \( m = n-10 = 1,2,\ldots 9 \) \hspace{1cm} (4)

For \( n = 21, 21,\ldots 39 \): \( (20+m)^2 = (20-m)^2 + 80m \) for \( m = n-20 = 1,2,\ldots 19 \) \hspace{1cm} (5)

For \( n = 41, 42,\ldots 64 \): \( (50+m)^2 = 2,500 + 100m + m^2 \) for \( m = n-50 = -9, -8,\ldots 14 \) \hspace{1cm} (6)

For \( n = 66, 67,\ldots 99 \): \( (50+m)^2 = (50-m)^2 + 200m \) for \( m = 16, 17,\ldots 49 \) \hspace{1cm} (7)

Reference:


Dr. Chichester completed B.S. and M/S. degrees in aeronautical engineering and a doctorate in electrical engineering, all three of which involved many courses in mathematics and its applications. While working as a math tutor for a tutoring team in a nearby school district he encountered some gifted students and began to study the patterns of mental mathematics.
Learning - Changing the Connections in the Brain

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Abstract
This article provides an extensive discussion of why creating neural association is vital to understanding and long term memory with recall. In the mathematics classroom, neural associations can be created by the proper and appropriate use of connections. But the timing of the connections is crucial; and they cannot be used on an irregular basis if the curriculum and pedagogy is to maximize the neural benefits. For example, they must be used at the beginning of a lesson, and the mathematical concepts or procedures desired to be connected must be simultaneously presented to the brain. This process will create a neural association. The article will give the readership food for thought and discussion because neural associations are used by the brain to help understand the mathematics being presented, and associations are used to store and recall long-term memories.

Neural Associations
Each long-term memory of individual concepts/situations/procedures/etc. (CSPE) is encoded in the brain through the firing of a neuronal circuit. As a new long-term memory is encoded, the dendrites of the neurons in the circuit change shape, which is how neuroscientists know a memory has been encoded. The circuit might require more than 15,000 synapses (McDermott, 2010). A synapse is formed at the intersection of a branch off the axon of one neuron and a dendrite of another neuron; it acts as a junction box where information is either passed on to the next neuron, or the flow is terminated. The number of synapses in a neuronal circuit required for a memory depends on the complexity of the memory. But the point is that every long-term memory of something learned is a neuronal circuit and when the related circuit fires, or discharges electricity and neurotransmitters (chemicals) from the initial synapse on to the last synapse; the memory is likely to come to consciousness in the form of recalling the CSPE.

When you simultaneously expose the brain to two or more individual and unique CSPE, it automatically creates a neural association, or physical connection, between the individual CSPE’s neuronal memory circuits (Schacter, 2001) (Eagleman, 2011) (McDermott, 2010). Creating neural associations between mathematical concepts/procedures and/or a simple real-
world contextual situation will increase the likeliness of being able to recall the mathematics given the situation. The more neural associations there are to, for example, factoring, the better the odds are of students remembering how to factor because long-term memory recall is processed through neural associations. “In the cell assembly, all neurons make synapses on all other neurons, so any part of the memory can trigger recall of the rest” (Seung, 2012, p. 72).

When you think of current curricular models and teaching practice, we argue that we are already making connections (neural associations) through the use of applications. This may be a true statement, but when you connect the “already learned” algebra to an application, the opportunity for learning the algebra has been lost. This is significant because encoding processes near the beginning of long-term memory formations are crucial to the durability of a memory (Schacter, 2001). Teachers can affect this memory process through the curriculum and pedagogy by using contextual situations at the beginning of a lesson to teach the algebra in the lesson and in the process create the neural association.

Creating neural associations seems so simple, but perhaps we need to know a little more about this universal neural behavior of creating associations. The creation of neural associations is automatic, which may not be good for learning algebra. For example if you are teaching students to factor trinomials for the first time using the pencil-and-paper method as you might have done 30 years ago, Luke (a student) smells spaghetti cooking from the cafeteria, and his brain may connect the neuronal circuitry being developed for factoring with the circuitry activated by the smell of spaghetti. At the same time, Jennifer (another student) hears the roar of a jet plane and connects the newly forming factoring circuitry with the circuitry activated by a jet engine roar. And then there is Rachel who is thinking about politics and connects the factoring circuitry to the Ronald Reagan circuitry. One can’t imagine that these connections would be useful to “learning.” And you would be correct because the automatic connections may not be mathematical in nature, and as such, will likely never be used in the classroom or curriculum. Fortunately, if these random connections are not fostered and used, they will be severed very quickly through the “use it or lose it” rule – a dominate function of brain behavior.

So why does the brain automatically create new connections between the neural circuitry of two distinct CSPE that are simultaneously exposed to the brain? The non-technical answer is that this is the mechanism the brain uses to processes long-term memory AND recall. An exception to associative memories is memory created by a strong emotion, which is instantaneous and long term. But without associations we could not store long-term memories or retrieve memories of say, how to factor trinomials since factoring may not have an emotional connection.

“Memory recall almost always follows a pathway of associations. … [W]e can only remember a few at any time and can only do so in a sequence of associations” (Hawkins, 2004, pp. 71, 73). One might question the validity of the science based on our own anecdotal thinking. That is, we usually don’t sense how we recall because it happens so quickly. However, age has a tendency to slow down brain processes so we become more aware of the recall process as older adults.

A Core Requirement for Learning

Neural associations are required for memory and learning (Edelman, 2006). We know how to create them in our students – simultaneously present the brain with two distinct CSPE. The problem with students
making their own neural associations is that teachers cannot use these seemingly random connections to create new memories (learning) of mathematical concepts and processes. The factor/spaghetti connection is not useful to anyone but one student, likewise for the factor/roar and factor/Reagan connections. Lynch & Granger (2008) make the point that “The more a particular set of connections are activated, the more they are strengthened, becoming increasingly reliable [for recall]” (p. 70). We might conjecture that Luke, Jennifer, and Rachel never re-used or experienced their connections more than once, so these connections will be severed soon after their creation. On the other hand, what if every algebraic concept or procedure is connected through a common theme running throughout the algebra curriculum, and these common associations were activated daily? With daily use of common theme connections, activating any one of the connected neural circuits will increase the likeliness of bringing the other circuits to consciousness (Hawkins, 2004) (Byrnes, 2001) (Seung, 2012). If you teach factoring through connections to zeros of polynomial functions, mentioning or using one will likely bring the other to consciousness. If factoring is also connected to the area of a rectangle, we have further enriched the associations increasing the chances of correct recall. (Beversdorf, personal email, 2003)

The depth and quantity of neural associations plays a crucial role in the ability to recall correctly in the long term (Schacter, 2001) (Thompson & Madigan, 2005). That is, upon the need to recall: “If one circuit fails to function, the other is likely to work” (Edelman, 2006, p. 33). Below is a teaching vignette that demonstrates connections. The lesson is preceded by an earlier lesson whereby students are given numerous real-world contextual data sets (functions in numeric form) with the directions to simply identify the shape of the graph and to conjecture when the relationship is increasing and when deceasing. I. V. drip data is included in the data sets as are a variety of other situations. Because the mathematics to be learned is presented as a simple real-world contextual situation, the mathematics becomes simpler to understand and can be learned faster than without a context as described by Greenspan & Shanker (2004, p. 241-2). This particular lesson is the first encounter with the symbolic form of a function. It will be connected to the graphic and numeric forms.

**Sample Lesson:**

A 1000 ml I.V. drip bag is attached to a patient; the nurse set the drip rate at 4 ml per minute.

$$L1 = \begin{array}{c|c|c|c}
0 & \text{-------} & \text{-------} \\
L2 & & \text{-------} \\
L2 \times 1 = 1000 & & \\
\end{array}$$

Question: If 0 minutes have passed since hanging the I.V. drip, how much fluid remains in the bag?

Student response: Since the bag has 1000 ml to start, at time 0 it must contain 1000 ml.

Why asked: The idea is to build a pattern so that students will generalize it. When students generalize, it forms a memory and establishes understanding. (Hawkins, 128, 89)
Question: If 1 minute has passed since starting the I.V. drip, how much fluid remains in the bag?
Student response: 996 ml.
How did you find that?
Student response: 1000 – 4.

Question: If 2 minutes have passed since starting the I.V. drip, how much fluid remains in the bag?
Student response: 992 ml.
Question: How did you find that?
Student response: (996 – 4).
Question: Ok, and where does the 996 come from?
Student response: (1000 – 4 – 4).
Why asked: To build a table like students have seen before and to build the generalized pattern of the data in numeric form.

Question: If 3 minutes have passed since starting the I.V. drip, how much fluid remains in the bag?
Typically, students answer 1000 – 4 – 4 – 4 because the average brain generalizes on the third iteration, but the teacher directs them to 1000 – 3×4. At this point, we will try another couple time values to confirm that we have the correct numeric generalization.

Once the numerical form of the model has been generalized, the next question to ask is how much fluid remains in the bag for \( t \) (or L1) minutes. Having taught this process in 50-75 classes, the author has never had a class not generalize at this point. There is no need to draw attention to the values in L1, nor are any other clues needed.

The power of symbols is demonstrated by asking how much fluid remains after 10 minutes, or 60 minutes, or maybe 120 minutes.

To make connections among representations, graph the data and the model.
Use trace and scroll along the data and then jump to the model for a variety of data points. Discuss increasing/decreasing. Trace to the zero (when the bag is empty). This will prime the students for a lesson on zeros at a later time.

The action of tracing on this model with a graphing calculator provides the simultaneous stimuli the brain needs to make connections among three representations of a function. In addition, this external stimulus connects the function representations to the real-world context giving the otherwise abstract mathematics a meaning (Pinker, 1997). When the bag is empty, the model has a zero of 250 minutes. The zero’s meaning is a significant emotional tag, and this will increase the likelihood of the memory surviving on a molecular level, provided the teacher discusses with the class the ramifications of an empty I.V. drip bag to the patient— or the nurse who hung the bag, or the doctor who prescribed the I.V., or the lawyer, etc. This one example may be insufficient for students to learn the connection and not have it severed, although it is considerably richer than the factor/Reagan connection. I use the words “may be” because you may have students who already have a strong emotional connection to an I.V., and for them this one encounter in a math class will likely secure the connection. The point is that we have connected mathematical concepts (variable, representation, rate of change, initial condition, etc.) to a known real-world context by teaching the concepts simultaneously – a requirement for the creation of neural associations.

Memory

It is obvious that humans do not recall all of our experiences or all of what we learned in math class. Memorization memory is especially bad with regards to recall over time (Langer, 1997) (Feynman, 1985). “… [I]t seems that a student actually can carry items for up to several weeks in working memory and then discard them when they serve no further purpose—in other words, after the student takes the test” (Sousa, 2010, p. 18). Memorization memory is typically created through practice. Further, it is common for the practice memory to be stored as working memory. Unfortunately, storing memorized mathematical content in working memory is not good for algebra students for two reasons: one is that working memory requires constant review (or repetition) to maintain the quality and recall, and two, working memory is commonly purged when the content is no longer of value (like after a midterm or other test). An athlete or musician uses practice to memorize a move or a movement. When practice subsides, the athletic move becomes rusty and the musical movement contains different notes or may be missing notes that were in the original memory. (Lynch & Granger, 2008, p. 70)

But then, this article is about the function of neural associations related to long-term memory with recall and has little to do with memorization memory. No matter what kind of memory, (memorization or associative-including emotional) we will lose the ability to recall over time. One wonders if your long-term memory is left to chance. It isn’t, but you may not have been aware of the association algorithm the brain uses to create and recall something learned in the long term.

Just like using neural associations at the beginning of a lesson, the use of
visualizations near the beginning of a lesson will improve long-term memory and enhance understanding of the concept or procedure in the lesson (Pinker, 1997) (Buonomano, 2011). Using a visualization to confirm pencil-and-paper work or teaching an application of the math already taught does not yield the same results for memory or understanding (Schacter, 2001). Another reason for using daily connections as a curricular goal, is that memory developed through the auto-associative memory system can be retrieved correctly even though you start with an incomplete version of it (Hawkins, 2004).

More than Meets the Eye

Another important benefit of using contextual situations to create neural associations is that the “…emotional stakes enable us all to understand certain concepts more quickly” (Greenspan & Shanker, 2004, p. 241). Greenspan and Shanker described teaching the concept of a tax as a portion of the whole while using pizza as their emotional stake; they also describe using candy to teach addition. The point is that the “emotional” context can be a simple real-world context that has meaning to students, like an I. V. drip. We also find that the use of neural associations provides a tool for mathematical understanding as indicated by Restak (2006), “We understand something new by relating [connecting] it to something we’ve known or experienced in the past” (p.164). The common practice of “explaining” does not provide the same level of understanding if explaining references no real-world context or makes no connections to a common theme or to previously learned mathematics.

Finally, we have one more interesting feature of using a common theme connection in all algebra lessons, and it is expressed well by McDermott (2010), “Suppose a novel experience requires the brain to build a network of, say fifteen thousand synapses. Each successive similar-but-somehow-different experience will require many fewer synapses (no one knows how many), simply because of the overlaps with the prior experiences. If … you constantly add to that store of experiences … the number of new synapses needed to encode information about a class of familiar objects (for example automobiles [or functions, equations, zeros]) might be quite small, perhaps hundreds compared to the original thousands of encodings” (p. 99). The implication is that reusing a concept or real-world contextual situation to teach new concepts requires less brain power as we progress through the common themed curriculum.

Final Observations

The bottom line of learning is that it changes brains. No question. Research from the neuroscience community is very clear; neural associations are the primary processes by which the brain stores and retrieves long term memories. In a topics-based curriculum we miss the opportunity to connect algebraic concepts or processes so that students can more likely recall and understand algebra. It is also the case that the brain needs to strengthen individual associations through repeated exposure to them in a variety of ways. For example the I.V. drip is used to make connections to variable, function, function representation, rate of change, initial condition, slope-intercept form, increasing, decreasing, equations, and zeros of functions. As the single common link contextual situation, it has the power to affect recall and understanding of considerable algebra. But more importantly, all of the related algebraic concepts can now trigger the recall of any of the other connected concepts.

If we think of the algebra curriculum as a series of “topics,” and we develop
activities for teaching each topic, we may disregard connections, simultaneity, early use of visualizations, contextual situations, pattern building, etc., thereby not capitalizing on core brain function. Teaching algebra as a series of topics suggests that the brain deals with this situation by memorizing concepts and processes using working memory. Working memory is not designed to act as long term memory and is often purged after a midterm exam (Sousa, 2010). Langer, (1997) agrees: “Memorization appears to be inefficient for long-term retention of information and it is usually undertaken for the purposes of evaluation by others” (p. 72).

As indicated before, the automatic, and seemingly random connections made by students do not last very long. And they are certainly useless to the teacher for teaching algebra because he/she does not know what they are. An algebra curriculum based on a common theme like function representation and function behaviors, solves this problem as nearly all algebraic concepts and procedures become directly connected – depending on the sequential order of the concepts being taught, the use of a graphing calculator (for processing function representation, pattern building, and analyzing function behaviors), and integration of simple contextual situations.

NOTE:

The references used in this article are mostly the work of neuroscientists, except Richard Feynman, who is a Noble Laureate in physics, and Gerald Edelman, who is a Noble Laureate in medicine. James Byrnes and David Sousa are educators, and Terry McDermott is a science writer.

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Informal Introduction to Limits: Providing students with a contextual and accessible problem to introduce the concept of a limit

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Abstract

Using a traditional “falling ladder problem” typically explored in differential calculus, Pre-calculus students and introductory calculus students can informally explore this and many other important concepts in calculus before they have learned proper analytic techniques. This article not only provides an investigation of the ladder problem, but outlines the necessary background knowledge, some possible student misconceptions and the meaningful mathematical connections that can be made amongst multiple representations of the problem. The article also provides justification and explanation as to how this type of problem serves as an opportunity for students to connect the mathematics content taught in high school and provides students with a meaningful foundation upon which important concepts in calculus and future mathematics can be built.
Informal Introduction to Limits: Laying foundations for calculus through pre-calculus problem exploration

Research has shown that when students are exposed to new mathematical situations and first encouraged to use their informal knowledge to make sense, they are able to assign meaning to mathematical concepts, symbols and procedures, deepen their understanding and make mathematical connections (Skouras, 2006). One way high school mathematics teachers can incorporate this informal knowledge, is through complex problem explorations. These types of investigations allow students to connect the mathematics content accumulated in school with their informal knowledge whilst constructing new knowledge and develop reasoning habits in the process (Graham, Cuoco, & Zimmerman, 2010). Depending on the type of problem chosen for exploration, these activities can also be used to prime students thinking for more advanced mathematical concepts and provide a meaningful foundation upon which these future concepts can be built (Kilpatrick, Swafford, & Findell, 2001). The problem exploration that follows not only discusses the different approaches students may take to solve a particular problem, but describes the mathematical connections that can be made, outlines the future mathematical concepts that it prepares for and demonstrates how students can connect their existing understanding of mathematics and with their informal knowledge to understand more advanced mathematics.

The Problem

Using a typical related rates problem from differential calculus, Pre-calculus students and introductory calculus students can informally explore many important concepts in calculus, namely, what it means to be a limit of a function.

A man is standing on the top of a 10 foot ladder that is leaning against your house. The base of the ladder begins to pull away from the house at a constant rate of 1 ft/s. Still leaning on the house, the top of the ladder begins to slide down towards the ground. What can you conjecture about the speed at which the man is falling just before the ladder hits the floor?

Note that at the Pre-Calculus level, students have not yet formally learned the analytical methods for determining the speed at which the man is falling. Often times, the analytical approach cannot be discussed until students learn differential calculus and a method for solving it may not be obvious.

Investigation

Initial Investigation

Before the teacher provides students with the function for the rate at which the top of the ladder is falling, students can conjecture about the rate using their existing understanding of the situation. Using their Pre-Calculus and Geometry background, students should be able to identify the right angle formed by the floor and the wall and, hence, determine the relationship between the height of the person off the ground and the distance from the base of the wall to the base of the ladder. (See Figure 1)
Using the Pythagorean theorem, students can rewrite the height of the person off the ground, $y$, in terms of the distance from the base of the wall to the base of the ladder, $x$. (See Figure 2) Students can then begin to conjecture about the speed at which the top of the ladder falls based on the information gathered about the position of the ladder at different times.

\[
x^2 + y^2 = 10^2
\]
\[
y^2 = 10^2 - x^2
\]
\[
y = \sqrt{10^2 - x^2}
\]

Figure 2

Initial Findings

By taking a numerical approach and testing different values for $x$, students can see the position of the ladder at different times and conclude that as the ladder moves further from the wall (values of $x$ get larger), the man gets closer to the ground (values for $y$ get smaller). In particular, students can observe that the maximum distance from the base of the wall to the base of the ladder is 10 feet. When $x = 10$ ft the ladder would have fallen completely and $y = 0$ ft.

More Advanced Findings

In more thorough numerical analyses, students can explore the change in the height as the distance, $x$, is changed incrementally. Using Microsoft Excel, or any other spreadsheet software, students can easily perform this numerical analysis on the position of the top of the ladder as it falls (See Figure 3).

From this table and graph, students can conclude that as $x$ increases incrementally, and the base of the ladder slides away at a constant rate, the distance that top of the ladder falls each time increases.

Guided Investigation

For more targeted exploration, the teacher can provide students with the function for the rate at which the top of the ladder falls. For this problem, the ladder falls at the rate determined by the function $f(x) = \frac{-x}{\sqrt{100 - x^2}}$, where $x$ is the distance from the base of the wall to the base of the ladder. Now applying their prior knowledge of functions, students can begin to make more concrete conclusions about the rate at which the ladder falls.
Numerical Analysis

Students can get a sense of the behavior of the function \( f(x) \) just by plugging in values for \( x \). As they do this numerical analysis, they will find that as they increase the value of \( x \), \( y \) values become increasingly negative. At this point, students will have to interpret the negative rate as an indication of the direction in which the person is moving. Thus, they can make the conclusion that as the base of the ladder slides further from the wall, the top of the ladder falls increasingly faster.

Graphical Analysis

At this point, it might help to consider using dynamic computer software to assist in exploration. A graph created using the interactive dynamic technology Geogebra is provided. (See Figure 4) From the graph, it becomes clear that as the ladder slides further from the wall and the \( x \) values approach 10, the \( y \) values become increasingly negative (with no bounds).

From this, students can return to the original question and note that as the ladder slides further from the wall, the rate at which the ladder falls becomes infinitely large, and hence falls faster and faster as it gets closer to the ground.

Mathematical Connections

Since this particular problem draws on the existing knowledge in many realms of mathematics, including trigonometry, algebra and function analysis, the opportunities for making mathematical connections are rich. Returning to the algebraic representation of the rational function \( f(x) \), students can connect their conclusions and understandings about the rate of the top of the ladder based on their numerical and graphical explorations to the form of the algebraic representation of the rate. For example, students can relate the undefined \( f(0) \) from their numerical approach and the vertical asymptotes at \( x = 10 \) and \( x = -10 \) to the \( \sqrt{100 - x^2} \) term in the rational function.

In addition, mathematical connections can be made across the multiple representations and methods of analysis employed. Dynamic technology can help students see these connections by providing them with an interactive environment that allows them to explore multiple representations of a function simultaneously. A multiple representation exploration of this problem done in Geogebra is provided.

(See Figure 5, on the next page)
In this particular exploration, numerical values for $f(x)$ at corresponding, incrementally increasing values of $x$ are extracted from the graph and stored in the spreadsheet adjacent to the graph using an animated slider. In this example, $x$ values are incremented by 0.1 ft, however, this value can be changed to tailor students investigations. On the left, the written function $f(x)$ and current point $(x, f(x))$ can be found in the algebraic view. In viewing these three representations simultaneously, students can formalize and solidify their understanding of rational functions, as well as, begin to analyze the behavior of the function and consider more advanced concepts. Moreover, introducing limits through a contextual problem allows students to connect mathematical processes and concepts with more intuitive and familiar situations, making it easier for students to conceptualize about the mathematics and abstract it into formal knowledge (NCTM, 2000).

**Laying Foundations**

Although Pre-Calculus students do not yet have the calculus background for solving this problem analytically, students can still answer questions concerning the limit of the function and make many meaningful connections along the way. An inquiry based activity like this allows students to review, apply and solidify their existing knowledge, lay foundations for future calculus topics. In particular, it provides an opportunity for students to develop their understanding of limits, relate limits to functional behaviors about asymptotes and generalize that if a function has a vertical asymptote and the limit of a function is evaluated at this asymptote, then the limit is either $\pm\infty$. This problem can also be used to explore several other important concepts regarding limits and other calculus topics. These include, but are not limited to, continuity, one side limits, domain and range, related rates and rate of change.
Background Knowledge

Though this problem does require some particular background knowledge from students, it allows for multiple entry points. Moreover, the multiple representation approach to this type of investigation makes it an accessible activity for all Pre-Calculus students. In particular, this problem does require student’s background knowledge of the Pythagorean Theorem and rational function. However, in general, students would just need experience working with functions, functional analysis and mathematical technologies.

Possible Misconceptions

Although this problem is accessible to pre-calculus students, students may have some misconceptions regarding the speed at which the top of the ladder falls. In particular, students may think that since the ladder slips away from the wall at a constant rate, it should also fall at a constant rate. Through further, more focused investigation, students can analyze their ideas about the rate at which the top of the ladder falls. To address this particular misconception, students can use dynamic software to gather more information about the change in the rate at which the top of the ladder falls. If students think the top of the ladder falls at a constant rate, then they should expect to find no change in the rate as the ladder slips away. However, through further numerical analysis, it becomes clear that this is not the case.

In Figure 6, we see the same Geogebra exploration as before, but with an added value in the spreadsheet view. Column C denotes the change in the rate at which the top of the ladder falls. From these values, we can see that as the base of the ladder slips further away from the wall and x approaches 10, the rate is actually increasing.

Other misconceptions could develop around the domain and range of the function in relation to the real life application of the problem. But as seen above, or this particular investigation, the dynamic technology has allowed students to quickly and easily test their conjectures about the mathematics involved. In this way, the activity and the mathematics involved allow students to work through these misconceptions on their own, ultimately allowing for construction and deep understanding of mathematical concepts.

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
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<td>8.9</td>
<td>-1.991923361794</td>
<td>0.0991891984</td>
</tr>
<tr>
<td>90</td>
<td>9.0</td>
<td>-2.0647416048</td>
<td>0.1128182431</td>
</tr>
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<td>-2.1948429694</td>
<td>0.1301013646</td>
</tr>
<tr>
<td>92</td>
<td>9.2</td>
<td>-2.3474276702</td>
<td>0.1525847008</td>
</tr>
<tr>
<td>93</td>
<td>9.3</td>
<td>-2.5302024623</td>
<td>0.1827747922</td>
</tr>
<tr>
<td>94</td>
<td>9.4</td>
<td>-2.7551887943</td>
<td>0.2249863319</td>
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</tr>
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<td>97</td>
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<td>99</td>
<td>9.9</td>
<td>-7.0179239296</td>
<td>2.0932386348</td>
</tr>
</tbody>
</table>

Figure 6
Conclusion

Throughout this activity, we have explored the benefits of using such a problem to informally introduce many key topics in calculus. This problem in particular, provides an entry into the concept of taking a limit and the properties surrounding special cases of limits. After completing the investigation, the ladder problem will provide students with a familiar, contextual mathematical scenario that to which they can relate important concepts to in the future. The teacher could continue with instruction on limits, including the formal definition of a limit, numerical, graphical and analytic methods for evaluating limits, the existence of limits, limit laws and related theorems about limits. As these topics are discussed, the teacher can relate back to the falling ladder example, allowing students to formalize their informal understanding of limits gathered during the initial activity.

As noted before, this problem exploration is just one example of the types of activities that Calculus and Pre-Calculus teachers can integrate into the classroom. The exact activity and structure of exploration used in the classroom should always be tailored to the particular group of students and the content they are learning. Problem explorations and authentic investigations provide students with meaningful learning opportunities that allow them to make sense of the mathematics for themselves. These types of experiences are invaluable to students and contribute greatly to the development and conceptual understanding of mathematics as a connected whole.
References


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Solving Ratio Problems Using Representations Listed in the Common Core

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Abstract: In the Common Core State Standards for Mathematics, standard 6.RP.3 requires students to solve ratio problems. It lists four representations that students can use when reasoning about ratios: tables of equivalent ratios, tape diagrams, double number line diagrams, or equations. Using equations (e.g., proportions) has been a predominant way that students in the United States have solved ratio problems. This article provides examples of how students can use the other three representations to solve ratio problems in meaningful ways.

Ratio and proportion are key mathematical concepts. Lest, Post, and Behr (1988) stressed that an understanding of ratio and proportion is critical because it is a foundation for mathematics courses in high school and beyond. The importance of these concepts is reflected in the fact that “Ratios and Proportional Relationships” is one of only five domains in the Common Core State Standards for Mathematics (CCSS-M) in both the sixth (p. 42) and seventh (p. 48) grades.

The CCSS-M, in accord with research that indicates a need for more extensive and in-depth instruction in ratio and proportion (e.g., Cramer, Post, & Currier, 1993; Heinz & Sterba-Boatwright, 2008; Lamon, 2012), suggests that students should learn how to use a variety of methods to represent and to solve problems involving ratios. Namely, standard 6.RP.3 states that students are to “use ratio and rate reasoning to solve real-world and mathematical problems, e.g., by reasoning about tables of equivalent ratios, tape diagrams, double number line diagrams, or equations” (p. 42). In the United States, setting up and solving proportions has been the predominant method taught to students for solving ratio problems. The purpose of this paper is to provide an introduction to the other three methods—ratio tables, tape diagrams, and double number line diagrams—for those not familiar with them.

Reasoning About Ratio Tables

A common ratio problem involves determining which of two pricing options is the better buy, such as the following: “Quicky Mart sells orange juice in a 20-ounce bottle for $1.80. Super Grocers charges $2.40 for a 32-ounce bottle of orange juice. Assuming that the two juices are of the same quality, which is the better buy?” A standard way of solving this type of problem is to find unit rates. However, if students are initially asked to use a pair of ratio tables to solve this problem, they have the opportunity to develop greater understandings of equivalent ratios and of how to compare two ratios. As such, standard 6.RP.3.a in the CCSS-M specifically states that students are to “use tables to compare ratios” (p. 42).

The two ratio tables, shown in Figure 1 below, are an illustration of how to use tables to compare ratios. The first row of data in each table contains the information given in the problem. Then, each consecutive row consists of two quantities that are in the same ratio as the given pair of quantities. For example, in the Quicky Mart table, all of the
pairs are equivalent to the ratio of 20 to $1.80. The goal is to find ratios so that either the number of ounces of orange juice or the price is the same in both tables. This is useful because if the price is the same, for example, then one can determine the better buy by comparing only the two amounts of orange juice that correspond to that price.

### Quicky Mart

<table>
<thead>
<tr>
<th>Ounces of Orange Juice</th>
<th>Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>$1.80</td>
</tr>
<tr>
<td>4</td>
<td>$0.36</td>
</tr>
<tr>
<td>80</td>
<td>$7.20</td>
</tr>
<tr>
<td>10</td>
<td>$0.90</td>
</tr>
<tr>
<td>5</td>
<td>$0.45</td>
</tr>
</tbody>
</table>

### Super Grocers

<table>
<thead>
<tr>
<th>Ounces of Orange Juice</th>
<th>Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>32</td>
<td>$2.40</td>
</tr>
<tr>
<td>4</td>
<td>$0.30</td>
</tr>
<tr>
<td>96</td>
<td>$7.20</td>
</tr>
<tr>
<td>16</td>
<td>$1.20</td>
</tr>
<tr>
<td>8</td>
<td>$0.60</td>
</tr>
<tr>
<td>4</td>
<td>$0.30</td>
</tr>
<tr>
<td>12</td>
<td>$0.90</td>
</tr>
</tbody>
</table>

Figure 1: Tables of Equivalent Ratios

The two tables in Figure 1 contain more rows than are needed to solve the problem so as to illustrate the various solution methods possible. For example, one student might generate the row in each table that has 4 ounces of orange juice by reasoning as follows: Looking at 20 and 32, I see that I can easily get the number of ounces in both tables to be 4 [the greatest common factor]. In the Quicky Mart table, I need to divide 20 by 5 to get 4, so I will also divide $1.80 by 5 and get $0.36. Then, in the Super Grocers table, I need to divide 32 by 8 to get 4, so I will divide $2.40 by 8 and get $0.30. Now I can see that the Super Grocers orange juice is the better buy because it is less expensive ($0.30 compared to $0.36) for the same amount (4 ounces) of orange juice.

Another student might look at the prices in each table and work on getting those two quantities to be the same. An efficient solution involves finding the least common multiple of $1.80 and $2.40, which is $7.20. With that information, the student uses a reasoning process analogous to the previous one to generate equivalent ratios. That is, knowing that $7.20 divided by $1.80 (Quicky Mart price) is 4, the student will multiply 20 ounces by 4 and get 80 ounces. Likewise, since $7.20 divided by $2.40 (Super Grocers price) is 3, the student will multiply 32 ounces by 3 and get 96 ounces. Then the student can compare the 80 ounces with the 96 ounces to determine that the Super Grocers orange juice is the better buy because one gets a larger number of ounces (96 compared to 80) for the same price ($7.20).

Note that in the first solution method, the better buy was the orange juice that had the lower price for the same amount of juice but in the second method, the better buy is the one that has the larger amount of juice for the same price. Being able to solve this problem using both methods (getting the price the same or getting the number of ounces the same) is an important component of an in-depth understanding of how to compare ratios.

Both of the solution methods just outlined were efficient in the sense that the student arrived at the target row of data (e.g., ratios in which one of the quantities was the same) in just one step. That is not always the case, nor is it necessary, especially when students are first learning how to compare ratios using tables of equivalent ratios. The rest of the rows in each table in Figure 1 were generated using an alternate solution method,
one based on halving, which is a method commonly used by students.

A student starting with the halving approach might reason about this problem as follows: I will take half of 20 and half of $1.80 to get 10 and $0.90. By taking half of 32 and half of $2.40, I get 16 and $1.20. I still don’t see how to get $0.90 and $1.20 the same, so I will take half of each quantity again to get 5 and $0.45 in the first table and then 8 and $0.60 in the second table. Now I see that if I add half of $0.60 to $0.60 that will give me $0.90, so then both tables will have a pair of ratios with $0.90 as the price. So, I will take half of 8 and get 4 and take half of $0.60 and get $0.30. Now I will add the corresponding quantities in those last two rows in the Super Grocers table to get 12 ounces and $0.90. I can see that Super Grocers is the better buy since 12 ounces for $0.90 is a better buy than 10 ounces for $0.90.

Note that this student did not ever need to use the 5 ounces for $0.45 row of data in the Quicky Mart table. Generating more data than is needed is a common occurrence for students learning to manipulate pairs of ratios to arrive at quantities useful for making comparisons. Early on, this is not a concern. In fact, this type of activity can help students develop understandings of how to generate equivalent ratios and of how to arrive at target quantities in increasingly sophisticated ways.

As mentioned previously, students can always divide to obtain unit ratios (e.g., divide each of 20 and $1.80 by 20 to get 1 ounce for $0.09 and divide each of 32 and $2.40 by 32 to get 1 ounce for $0.075). However, it is not necessarily useful to promote this approach initially for several reasons, two of which are that (a) doing so would shortcut students’ spontaneous solutions methods which provide the opportunity for students to develop deeper understandings, and (b) there are often more efficient and more elegant methods than finding unit rates, such as the methods just discussed. Teaching students to look for and carry out efficient and elegant solution methods helps students develop a powerful number sense, including an ability to use numbers in flexible and efficient ways.

Reasoning About a Tape Diagram

An example of another typical type of ratio problem that students need to know how to solve is the following: “A fruit punch is made by mixing juice and ginger ale in a ratio of 5:3. For her party, Sarah wants to make 28 cups (7 quarts) of punch. How many cups of juice and how many cups of ginger ale does she need?” Tape (or strip) diagrams are a useful way to solve this problem because they help students coordinate their accumulation of cups of juice and ginger ale until they arrive at their target amount of punch.

Students would begin solving this problem by drawing tape diagrams to represent the fact that the punch is made by mixing juice and ginger ale in the ratio of 5:3. (See Figure 2a.) At this point, it is not necessary to assign a unit (e.g., cups) to each rectangle in the strips. As long as each rectangle is understood to be the same size, the 5:3 relationship is represented. Then, as students focus on making a total of 28 cups, it is useful to think of each rectangle as representing one cup.

After making an initial drawing of strip diagrams (e.g., Figure 2a), students’ solution methods can vary. Next, I elaborate on two solution methods: one that is rather rudimentary and a second one that is more advanced.

juice: 

| | | | | | | |

ginger ale: 

| | | | |

Figure 2a: Original Strip Diagram

A rudimentary solution method, one that is an important starting point for many students, is to continue to make drawings of
the strip diagrams until the target goal of a total of 28 cups is reached. The student understands that 3 cups of ginger ale will be added for every 5 cups of juice that is added. With this understanding, the student accumulates three sets of the 5:3 pair of strips (See Figure 2b.), noting that four pairs would produce 32 cups of punch, which would be too much.

<table>
<thead>
<tr>
<th>juice:</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>ginger ale:</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Figure 2b: Accumulating Juice and Ginger Ale**

Now the student needs to determine how to get from the 24 cups already accumulated to the 28 cups that is needed. Somehow, four more cups of the 5:3 ratio must be added. By accumulating the 5:3 pairs of strips, and keeping track of the total accumulated along the way (i.e., 8, 16, 24, 32—which was too much), the student is likely to notice that 8 more cups of punch are made each time a set of the 5:3 combination is added. So, if 4 cups are needed, that means that half of the 5:3 combination is needed and therefore half of each ingredient is needed. The student can then partition the 5:3 pair of strips into a 2.5:1.5 pair of strips, producing a strip diagram like the one in Figure 2c. From the diagram, the student can determine that 17.5 cups of juice (3 x 5 + 2.5) and 10.5 cups of ginger ale (3 x 3 + 1.5) are needed to make 28 cups of fruit punch.

<table>
<thead>
<tr>
<th>juice:</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>ginger ale:</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Figure 2c: Final Strip Diagram**

As students progress, they will be able to solve this type of problem in more efficient ways. For example, after drawing the original strip diagram, they may notice that the total of 8 cups of punch is a key quantity. They then divide 28 cups (the target) by 8 to determine that 3.5 of the 5:3 pair is needed. Thus, 17.5 cups of juice (3.5 x 5) and 10.5 cups of ginger ale (3.5 x 3) are needed to make 28 cups of punch.

**Reasoning About a Double Number Line**

Another type of ratio problem that students need to be able to solve is one that involves constant speed: “If Diamond can read 30 pages of her book in 45 minutes, how long will it take her to read 100 pages, assuming she reads at a constant rate?” A double number line is a useful representation for students to use to solve this type of problem because it can help them coordinate the changes in both the number of pages and the number of minutes, while keeping the ratio between those two quantities the same.

To solve this problem using a double number line, students would label one number line as “Number of Pages” and the second number line, beneath the first, as “Number of Minutes.” Then, students would make hash marks for 30 pages and 45 minutes, lining them up vertically and labeling them accordingly. From that point on, students can add pairs of points, always aligning them vertically, to represent other ratios that are equivalent to the original ratio of 30 pages to 45 minutes.

The double number line diagram illustrated in Figure 3, shown below, could be used as the basis for the following reasoning:

I know if I double the 30 and double the 45, I will get an equivalent ratio, so I will plot 60 pages to 90 minutes on the number lines. I will then add another 30 to the 60 and another 45 to the 90 to get 90 pages to 135 minutes.

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minutes. Next, I need to add only 10, not 30, to the 90 to get 100. So, I need to determine the number of minutes that corresponds to 10 pages. Going back to the beginning of the number lines, I see that if I divide 30 by 3, I will get 10. So, I will also divide 45 by 3 to get 15 and plot the 10 pages to 15 minutes ratio. Now, I can add 10 to 90 to get my target value of 100 pages and then add 15 to 135 to get 150 minutes, which is the answer to the question posed in the problem. That is, it will take Diamond 150 minutes to read 100 pages.

![Double Number Line](image)

**Figure 3: Double Number Line**

**Conclusion**

Ratio tables, tape diagrams, and double number lines are three representations that students can use to reason about ratio problems. Promoting ratio and rate reasoning is the essence of standard 6.RP.3. Asking students to solve ratio problems by reasoning about these three representations provides them with the opportunity to develop deeper understandings of how to use ratios and proportions in order to solve problems in flexible and meaningful ways.

**References**


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