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Call for Manuscripts
The New Proposed Standards in Math Education

In early February the New Jersey Department of Education came out with a proposed revision of the New Jersey Core Curriculum Content Standards in mathematics (CCCS-Math). Most of you know that the original CCCS-Math came out in 1996, and then revised in 2003. Those previous two versions were very forward-thinking, and reflected a great deal of consideration on the part of many groups in the state. Specifically, mathematicians, mathematics educators, school leaders, and parents all had a hand in the final document in 1996, and its revision in 2003. The CCCS-Math received high marks from several sources (Achieve, Inc. in 2004 and The National Assessment of Educational Progress in 2007, among others).

The CCCS-Math achieved a wonderful balance between conceptual knowledge and procedural skills. Conceptual knowledge is the understanding of the underlying principles of mathematics; the foundational ideas of a topic. Students need to have conceptual knowledge in order to build the foundation of mathematical knowledge. As well, students need procedural skills. Procedural skills are the rules and algorithms of solving a mathematical problem, or understanding a mathematical concept (for instance, the algorithm for adding two digit numbers). Students need to be able to do this and other calculations without a calculator or computer to solidify their understandings of this and other mathematical concepts. The key word here is balance. It’s very important that students be able to carry out many of the most fundamental mathematical calculations on paper without the use of a calculator (addition, subtraction, multiplication, and division of whole numbers, fractions, and decimals, symbolic manipulation of expressions, among other calculations). It is also very important that students understand the fundamental underpinnings of mathematics (concepts such as ratio, fraction, and place value, to name but a few). Understanding these, and connecting them to other concepts, builds in the students’ minds a “web” of understandings and thus these ideas are easier to retrieve by the students.

Sadly, the (proposed) newly revised CCCS-Math will drastically de-emphasize the importance of conceptual understandings and increase the importance of procedural skills. They will eliminate calculator use below the 8th grade. They will eliminate the use of manipulatives in the classroom. They will drastically weaken the process standards (Problem solving, Reasoning and Proof, Communication, Connections, and Representation).

It is vitally important for everyone here in New Jersey to understand for themselves the issues, and the arguments (both pro and con) in this debate. The New Jersey website that contains the (proposed) revised standards can be found at: www.state.nj.us/education/aps

For a view against the (proposed) newly revised CCCS-Math, go to: http://sites.google.com/site/cmeofnj. For a view in support of the (proposed) newly revised CCCS-Math, go to: http://njworldclassmath.webs.com.

As always, to contact me, email me at tpwalsh@kean.edu.
Reflecting Upon a Special (Education) Event: A Conversation with Counsel

John E. Hammett III, Ed.D., Associate Professor, Saint Peter’s College

On December 3, 2008, over 500 educators gathered in Jamesburg, N.J., for the Association of Mathematics Teachers of New Jersey's First Annual Special Education/Mathematics Conference, with a theme of Preparing for State Standards & Assessments. This article provides a glimpse into this conference through an informal discussion on the role of the regular education teacher in special education and common errors with IEPs.

Introduction

Keynote addresses by several officers from the New Jersey Department of Education formed the figurative bookends to the conference program. In the morning, Roberta Wohle, Director of the Office of Special Education, and Peggy McDonald, Manager of the Office of Special Education Programs in the Bureau of Program Accountability, set the stage for the day's informative discussions; they helped conference attendees comprehend the implications of the mandates of the legislation contained in the Individuals with Disabilities Education Act (IDEA) and No Child Left behind (NCLB) on math instruction, programming, and services for students with disabilities. Statistics were shared that dramatically highlighted measured gaps in achievement between general population students and students with disabilities. In the afternoon, Robert Riehs, Mathematics Specialist and the 2008 New Jersey Outstanding Mathematics Educator as recognized with the Association's distinguished Max Sobel Award, closed the conference by advocating that students with mild disabilities can understand mathematics without implementing some accommodations. Bob stressed that making accommodations in such cases may unfortunately preclude these students from constructing their own more meaningful conceptions of mathematics. Another highlight of the conference was the presence of and presentation by National Council of Teachers of Mathematics Past President Lee V. Stiff, Professor of Mathematics at North Carolina State University. Lee's lively, congenial, and yet candid talk focused on how the tracking of math students impacts these learners in remarkable, even dramatic, ways that stretch far beyond the classroom.

The breakout sessions offered a plethora of professional development opportunities. The speakers not only included Past Presidents of the Association, such as Paul Lawrence, Joan Vas, Agnes Azzolino, and Deborah Ives, but also Sandy Shields, a Preschool Inclusion Teacher from the Dumont Public Schools, who was a 2008 recipient of the Fellowship Program in Inclusive Education sponsored by the New Jersey Council of Developmental Disabilities.

One session addressed the conference themes somewhat uniquely, namely from a legal perspective. Anthony Sciarillo, Chairperson of the Department of Education and Director of Graduate Education at Saint Peter's College, is also an attorney at law with several decades worth of combined legal and educational practice. As Past President of the Association, I spent part of a recent evening chatting with Tony about the conference. To provide a sample of the conference experience, here are his reflections, micro-proceedings if you will, on his session entitled, “The Role of the Regular Educator in Special Education and
the Ten Most Common Errors Made in the IEP Process,” and on the event as a whole.

**What issues have you observed regular education math teachers encountering with regard to Special Education and IEPs?**

“One of the biggest issues for math teachers is that they may get asked to sit in on IEP [i.e., Individualized Education Plan] meetings simply because they are math teachers at their school, even though they may not personally know the students being discussed and may not have had them in their own classrooms. The challenge here is that these teachers need to provide substantive information without necessarily having first-hand experience. If the regular education math teachers don't have the information, they need to get it.

*Here's why.* The most critical aspect for regular math classroom teachers in this situation is the ability to plan for the next year's IEP. To do this, teachers need to be prepared to make three observations about the special education student: first, determine the student's current level of educational performance; second, compare the student's current level of mastery -- how did the student do on the task at hand -- against what is called for or expected of the student in the IEP; and third, review the stated goals and objectives in the student's current IEP so that modifications can be proposed for the next year's plan.

Here's a classic example. A special education student currently can complete 5 out of 10 multiplication problems successfully. The student's IEP says that this student is expected to complete 7 or 8 out of 10 successfully. The student appeared to miss the target. Why? That needs to be determined before the next year's IEP can be modified or the goals tweaked in some way. For instance, increasing the expectation for the student to complete 8 or 9 out of 10 correctly may seem automatic; however, the better choice for this student may be to keep the goal at 7 or 8 out of 10. The regular and special education teachers need to work together to tweak the goals and objectives. In fact what the teachers really need to do is identify reasons why the goal wasn't met, and address them. As a result, some modifications may be needed, without changing the goal.

And here are two related concerns. Sometimes a curriculum doesn’t get completely modified before it is actually implemented, and sometimes a curriculum does indeed get modified but it doesn’t get implemented properly. Yes, making these required changes can be time-consuming. However, the curriculum does need to get modified completely and then that modified curriculum has to be adequately implemented in order to assure that the Special Education student has the opportunity to satisfy the goals and objectives stated in the IEP.”

**What role does professional development play in helping regular education math teachers to become better equipped to work more effectively with their Special Education students?**

“When math teachers at the same grade level plan, develop, and implement a curriculum together, they should also work with other teachers in grade sequence [i.e., those teaching at the grades below and above their level] to tie together a transition plan for Special Education students. The regular education teachers need to know what is essential, what is at the core of their curriculum for their grade level to assure
that Special Education students at the very least learn that content, even if in modified form, to be ready for the next grade level. For instance, what essential concepts do Special Education students need to learn in fifth grade in order to be successful in sixth grade? What all this boils down to is increased grade level articulation.

Teachers should have common prep periods in addition to their individual prep periods for this increased planning at grade level and beyond. Preferably, having two or three grade levels together would be ideal. Yes, it's expensive, especially in these economic times, because it's paid for by their boards of education as additional instructional expenses. Every time a teacher is pulled out of a classroom, even for planning purposes, there's a cost. Arguably, for every 20 periods or so a teacher is not in class, another teacher is needed. Those actions cost the district an extra teacher's salary plus benefits. This isn't an easy sell to school districts with limited resources; they don't readily see this investment as translatable into student achievement.

There's a catch here. When a district gets more students, they can demonstrate the need for hiring teachers; unfortunately, just saying that hiring more teachers will allow them to improve the delivery of educational services won't work. There's a definite tension between improving instruction and sticking within the budgetary cap of 4% on revenue. If districts want test scores to go up, they need more teachers and more professional development. Districts want to deliver their services in the most effective and efficient manner, but that costs time and money. This would require that their teachers actually teach fewer classes for a finite period time as they plan and make these changes.

Do you have any gentle reminders about working with IEPs for those of us who are regular education math teachers and have limited experience working with Special Education students?

"We always need to remember that an IEP establishes an educational contract between the school district and the student; as teachers, we have both a professional and a legal obligation to work to modify the curriculum, and then implement those modifications so to maximize the Special Education student's opportunities to achieve meaningful educational progress. And don't forget that modifications don't mean just cutting back on quantity, for example expecting only 50 out of 100 problems solved by Special Education students instead of all 100 problems solved by regular education students. Modifications are much more than that. They should be about what these students need to be successful both in and out of school.

The success of a classroom with both a regular education math teacher and a Special Education teacher really is a reflection of the professionalism of the two teachers working well together. I've noticed that recent graduates from teacher education programs have a good sense of how this works, perhaps because they've been exposed to this dynamic already in their education classes."

Since mathematics teaching should foster meaningful learning of math concepts, regular education math teachers should relate well to this objective of meaningful progress for Students with Special Needs. In closing, what were your overall impressions of the Association's First Annual Special Education Conference in
December 2008, and will you be speaking at the Second Annual in December 2009?

"The conference experience was excellent; I really enjoyed it. The pace [of the program] was good, the conference was well structured, and the variety was overwhelming! The participants were relaxed enough to talk amongst themselves and with the presenters. It was good that they had the opportunity to talk to one another. There were lots of formal and informal conversations. The teachers attending the conference were enthusiastic and interested throughout the day, and they seemed to leave [at the end of the conference] as engaged in the discussions as when they arrived. All these indicators told me the conference was a huge success. And I got to meet lots of new people. It was great to meet new teachers, since I mainly interact now with administrators in my position at the College. And yes, I'll be there [in Jamesburg on December 1, 2009]. I'm really looking forward to it!"

Great! See you in Jamesburg, Tony.
A Delightful Inquiry Problem for Geometry Students of All Levels
Ray Siegrist, Ed. D., Assistant Professor, SUNY College at Oneonta

In this article, the author provides a multi-grade level approach to a basic geometry problem. The problem focuses on inscribing circles within an equilateral triangle. A variety of creative inquiries to the problem are explored with ideas presented for use in the mathematics classroom.

Recently an interesting problem (Dunn, 1980) came to my attention that can be used in an inquiry into geometry. The problem is as follows:

*If a circle of radius one is inscribed in an equilateral triangle, what is the area of the triangle?*

The problem can be done in parts to build a nice inquiry; in other words, the problem is complex or takes several steps for solution. It can be adapted for use in an elementary, middle, or high school classroom. Further, several formulas are used in a problem-solving situation. The problem also has the potential for stimulating rich dialogue in a cooperative problem-solving environment.

Furthermore, many of the standards promoted within the Principles and Standards for School Mathematics (National Council of the Teachers of Mathematics, 2000) are highlighted by the stated problem. First, the problem addresses the NCTM geometry standard by asking for an analysis of two-dimensional figures. Second, by inquiry into what happens when circles tangent to the triangle and original circle are added in stages, students are meeting the NCTM algebra standards. Collecting data and making conjectures about patterns also meet reasoning and proof standards within algebra. Third, since this is a rich problem, students meet the NCTM problem-solving standard by engaging in genuine problem solving. Finally, cooperative groups of students meet the NCTM communication standard by generating strategies for solving this problem through mathematical dialogue.

**Approach for Elementary Students**

Students start by making an equilateral triangle from a square with eight-inch sides (Figure 1). (It is best to have a square or triangle as large as possible.) Then, a line is drawn from the midpoint of the top side of the square to the midpoint of the bottom side of the square. Finally, from each of the vertices of the side where the midpoint was first selected, an eight-inch line is drawn to the constructed midpoint line. In fact, measure eight inches from the vertex to the line. The result is an equilateral triangle with eight-inch sides with an altitude-median-angle bisector line. The directions can be given orally or in written form to see how well students can follow directions. This is an excellent opportunity to introduce or reinforce geometric vocabulary.

**Figure 1**
*Drawing an eight-inch equilateral triangle.*

Students draw the other altitude-median-angle bisector lines. Since the altitude-median-angle bisector lines intersect in a common point, students may well have
a “wow” experience. A side inquiry can question what happens if the triangle is not equilateral; how the altitude, median, and angle bisector constructed; and what is the significance of their intersections. Then, students construct an inscribed circle using a compass (at the intersection of the midpoints of the legs of the triangle). (See Figure 2). At this point ask, can other circles tangent to the circle and triangle be drawn? Stage Two (Figure 3) and Stage Three (Figure 4) are possible answers. The pattern of the number of circles added at each stage can be discovered.

In a second activity, students can be given the Stage Three (Figure 4). Ask students to make conjectures about what they see. Students might have to be prompted to speculate on the figure that preceded the one given them (Figure 3). An interesting discussion with opportunity to introduce and reinforce geometric vocabulary can lead to a discovery of the pattern of the number of circles added each time. Since self-similarity is present, Figure 3 represents a fractal, and some students will recognize the fractal pattern.

**Approach for Middle and High School Students**

A circle of radius one is inscribed in an equilateral triangle (Figure 2). Questions that might be posed include: What is the area of the triangle? What is the circumference of the circle? What is the space within the triangle that is not covered by the circle?

Encourage students to develop a chart to organize their data as they answer questions (Table 1). The area of the triangle is $3\sqrt{3}$ by the following argument. A line segment drawn from a vertex of the triangle to the center of the circle bisects an angle of the equilateral triangle. Therefore, the triangle created using this segment and by drawing a radius from the center of the circle to a point of tangency with the triangle is a 30-60-90 triangle. Since the radius, side opposite the thirty-degree angle, is one, the angle bisecting segment, the hypotenuse, is two. This makes the height of the larger triangle three. By constructing the height of the triangle, another 30-60-90 triangle is created with sides measuring $2\sqrt{3}$ making the area of the large triangle $3\sqrt{3}$.

Two calculations produce data that eventually leads to interesting results. The circumference of the circle is $2\pi$. The space within the triangle not occupied by the circle is $3\sqrt{3} – \pi \approx 2.05$. As circles are added to the figure, students calculate the sum of the circumferences of the added circles and the area of the triangle not covered by any circle.
After three circles are drawn tangent to the circle and triangle of Figure 2 in the spaces at each vertex (Figure 3), another set of questions can be posed: What are the radius and the circumference of the newly formed circles? What is the sum of the circumferences of the newly formed circles? What is the space of the triangle not taken up by all of the circles?

Suggesting students construct a triangle about one of the newly added circles might aid in solving the problem. Referring to previous work on the problem, the height of the triangle is one. The constructed side is parallel to the base of the large triangle; therefore, the smaller triangle is similar to the larger triangle with a scale of 3:1.

After adding three circles around each of the three circles most recently added (Figure 4), students answer the last set of questions once more. I suggest that students create their own chart. Table 1 is provided as a reference for comparison of the answers reached by student consensus. As the added circles in Stage 4 are rather small, further entries to Table 1 can be obtained by a combination of pattern recognition and calculation, if needed.

A “wow” result is the sum of the circumferences of the circles added at each stage is constant at $2\pi$. A final question that has to be asked: At what stage do the circles cover the entire area of the triangle? Area of the circles = $\pi + 3(1/9\pi) + 9(1/81\pi) + \ldots = \pi(1 + 1/3 + 1/9 + \ldots)$. The sum of the infinite geometric series is $1/(1 – 1/3) = 3/2$; therefore, the area of the circles equals $3/2\pi$ or approximately 4.7 square units. But, the area of the triangle minus the area of the circles $(5.2 – 4.7 = .5)$ is approximately .5 square units of space uncovered. Even when the process is applied an infinite amount of times, the smallest circles cannot fill in all of the gaps. This is another “wow” result.

<table>
<thead>
<tr>
<th># of circles added</th>
<th>Height of triangle</th>
<th>Circumference of added circles</th>
<th>Area not covered</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>$2\pi$</td>
<td>$5.2 – 3.1 = 2.1$</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>$2\pi$</td>
<td>$5.2 – 4.2 = 1.0$</td>
</tr>
<tr>
<td>9</td>
<td>1/3</td>
<td>$2\pi$</td>
<td>$5.2 – 4.5 = 0.7$</td>
</tr>
</tbody>
</table>

### Table 1

*Data Collection Chart*

To continue Table 1 to Stage $n$, the number of circles added equals $3^{n-1}$ and the height of the triangle $3^{2-n}$. The sum of a geometric series, $(r^n a_n)/(r – 1)$, can be used to obtain the sums of the areas of the circles.

$$\pi((1/3) 3^{1-n} – 1)/(1/3 – 1) = \pi(-1/2) 3^{1-n} + 3/2\pi.$$  

For Stage 3 (the last listed in Table 1), $3^2 = 9$ circles added. $3^{1} = 1/3$ is the height of the triangle drawn about an added circle.

$$\pi(-1/2)3^2 + 3/2\pi = \pi(-1/18 + 3/2) = \pi(-1/18 + 27/18) = 26/18\pi = 13/9\pi$$

is the sum of the areas of the circles.
This problem is complex; therefore, proper questioning and the discreet hint will not give away the entire solution. Students with the prior knowledge of equilateral triangles, 30-60-90 triangles, and the area of triangles can be kept engaged in problem solving. An opportunity is present to apply the formulas for the sum of a geometric series and the sum of an infinite geometric series in an actual problem. A rare occurrence is possibility hearing students exclaim, “Wow!”

References


This article discusses the use of paradoxes in the classroom. Connected to logic, geometry, and arithmetic, paradoxes provide rich ground for discussion in the mathematics classroom. The author discusses several classic paradoxes from a variety of sources as well as guidance to the classroom teacher wishing to use them.

An unusual topic that can be taught in a mathematics classroom that connects arithmetic with geometry and logical thinking with literature is a paradox. A paradox can be an apparently true statement or group of statements that leads to a contradiction or a geometric situation which defies reasoning (Danesi, 2004). A paradox can be in the form of a geometric figure which causes a contradiction or optical illusion (Danesi, 2004). The newly adopted 2008 New Jersey Core Curriculum Content Standards for Mathematics Standard No. 4.2.2A and Standard No. 4.5 state that students must be able to solve real problems, reason effectively, and make logical connections. Students need more instructional time for problem solving and active learning. One of the ways students can solve logic problems is by studying paradoxes. Paradoxes help students: comprehend geometric properties; understand and transform shapes; recognize, describe, extend, and create space-filling patterns.

In literature, philosophy, and logic there are paradoxical statements that cause a contradiction, some of which will be mentioned in this paper. For example, one real life situation is Bertrand Russell’s the Barber Paradox (See Figure 1). This type of paradox is an example of a liar paradox (Danesi, 2004).

Suppose you walk past a barbershop and see a sign that says, “Do you shave yourself? If not, come in and I'll shave you! I shave anyone who does not shave himself, and no-one else” (Joyce, 2002, p.1). This is known as the Barber Paradox. The question is did the barber shave himself?

Another example of a liar paradox in literature is from Danesi: “This sentence is false.” or “The next sentence is false. The previous sentence is true.” (2004, p. 144-146)
These statements are paradoxical because there is no way to assign them to a consistent truth-value. Consider that if “This statement is false” is true, then what it says is the case; but what it says is that it is false; hence it is false. On the other hand, if it is false, then what it says is not the case; thus, since it says that it is false, it must be true. This is to be distinguished from the common colloquial expression “I tell a lie” when the speaker has realized that he has just accidentally told an untruth.

Within the topics of mathematics, there are many geometric paradoxes. The mathematician and philosopher, Martin Gardner, describes many different paradoxes in his books of puzzles (1959, 1988). Sam Lloyd (2005) introduced many magic inducing puzzle gadgets during his lifetime. Sam Loyd Jr. compiled the most comprehensive collection of his father’s puzzles in the Cyclopedia of Tricks and Puzzles published in 1915. One of his famous geometric puzzles is “The Extra Square”. (See Figure 2 shown below.) It is an 8 x 8 square, with an area of 64, divided into two congruent trapezoids and two congruent triangles. The four pieces are cut apart and rearranged into a 5 x 13 rectangular grid. The area of this new rectangle is 65. The question is where did the extra square come from?

Another mathematical paradox using area similar to Loyd’s “The Extra Square” problem is William Hooper’s paradox called “Geometric Money” (Bogomolny, 1996). Shown in Figure 3, this paradox begins with a 3 x 10 rectangle that is cut into two equal triangles and two equal trapezoids. These four pieces are cut apart and rearranged into a compound figure putting the two triangles together and the two trapezoids together. The 30 square unit grid is transformed into a 32 square unit grid. Putting the original rectangle next to the composite figure shows that the diagonals of the figures have the same slope. The question is where did the two square units come from?

Yet another famous Sam Lloyd paradox is the “Get Off the Earth”
puzzle, invented in 1898 (Danesi, 2004). The puzzle shows a circle representing the globe with 13 Chinese warriors around the rim. As shown, in Figure 4, the earth disk cut from its original position. When the disk is placed inside and the arrow is pointing N.E. you can count the 13 Chinese warriors. When the disk is rotated left so the arrow points N.W., a warrior disappears to make the count 12. The question is where did the 13th warrior go?

There are many Sam Lloyd puzzles like “Get Off the Earth” one of these is called “Count the Monkeys” (Education World, 2006). It is similar to the Chinese warriors, but the circle has monkeys instead. By rotating the disk in the center of the circle, the number of monkeys changes from 13 to 12. (See Figure 5 below.)

Figure 4


Figure 5

Count the Monkeys (Education World, 2006). Reprinted with permission.

In each type of paradox mentioned, there is mathematical vocabulary and problems to be solved. Each kind of paradox, whether it is in literature or mathematics, requires the student to apply critical thinking skills to understand the problem and figure out the answer. These paradoxes can be integrated easily into a K-12 curriculum. With further exploration, the classroom teacher can find many additional
paradoxes by referring to the sources cited in the reference section.

References


New Jersey Mathematics Teacher, May 2009, Volume 67, Issue 1 14
This article focuses on the application of parametric equations. The author presents an approach to Hungerford’s original models, adapted for use in the algebra and precalculus classroom.

Parametric equations are very useful for representing graphs of curves that cannot otherwise be expressed as functions that define y in terms of x, as well as for modeling real-life situations involving motion of an object along a given path by providing the coordinates of positions (x, y) of the object over time (t). The Algebra Standard of Principles and Standards for School Mathematics (NCTM, 2000) suggests that mathematics curricula for grades 9-12 include the use of “a variety of symbolic representations, including recursive and parametric equations, for functions and relations” (p. 296). However, in spite of the usefulness and the incorporation in the Standards just described, in my personal teaching experience I have found that students are generally weak in their understanding of parametric equations. In an effort to share some ways that I have found to make parametric equations more approachable to algebra and precalculus students, I have described some problems adapted from those presented in a precalculus textbook (Hungerford, 2004) that I use when introducing parametric equations using a TI-83/84 graphing calculator in a previous article (Herman, 2006). While basic information is given in my earlier article, this article is an extension of the use of parametrics involving physics applications of two sets of parametric equations. The problem presented below is also adapted from the precalculus textbook (Hungerford, 2004). I add a description of how I use the problem in class, along with instructions for using the TI-83/84 calculator for in-depth analysis within the discussion of the problem.

The Problem with Hungerford’s Original Models

Consider a ball in flight in time t seconds as modeled by:

\[
\begin{align*}
  x &= 62 \cos (68.75) t \\
  y &= 5 + 62 \sin (68.75) t - 16t^2.
\end{align*}
\]

Also consider a person on a Ferris Wheel, with this person's Ferris Wheel ride modeled by:

\[
\begin{align*}
  x &= 20 \cos (40.11 t) + 100 \\
  y &= 20 \sin (40.11 t) + 20.
\end{align*}
\]

These models involve degree measures (Hungerford, 2004, uses radians), and all linear measurements are made in feet. The mode and window settings in Figure 1 provide the given static graph of the situation. Pressing the \( \boxed{X,T,\theta,n} \) button which normally produces X in function mode will automatically produce the appropriate T in parametric mode. My students also like the graphing feature that looks like a moving object leaving a trail (i.e., as shown in Figure 1), which can be selected by pressing \( \boxed{ENTER} \) to cycle through graphing options to the left of the x-component of the parametric equations.
Rather than providing the viewing window in Figure 1, I ask students to find a good window that will enable them to watch 9 seconds of flight time so that the parabolic path of the ball and the person’s Ferris Wheel ride are both visible on the same screen. Squaring (ZOOM > 5:ZSquare) the window helps to make the Ferris Wheel ride appear circular rather than elliptical.

I review the physical interpretation of the equations with my students, arbitrarily placing a person at position (0, 0) to throw the ball towards the Ferris Wheel. When the ball is thrown, it first leaves the person's hand at an initial height of 5 feet above the ground with initial velocity 62 ft/sec at an angle of 68.75° with the horizontal. The negative 16 in the equation for the \( y \)-coordinate of the ball in flight accounts for acceleration due to gravity (with an acceleration constant of -32 ft/sec\(^2\) on Earth when the second derivative of -16t\(^2\) of the \( y \)-position equation is taken). Figure 2 depicts this information.

According to the given equations, the person throwing the ball from position (0, 0) is 100 feet in the horizontal direction (along the ground) from the bottom of the Ferris Wheel, (100, 0). The ball is thrown when the person on the Ferris Wheel is at the 3

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**Figure 1**

*Sample settings for viewing ball in flight and Ferris Wheel ride using original models from Hungerford, 2004.*

**Figure 2**

*Initial conditions of ball in flight using original model.*
The obvious question arises: Can the person on the Ferris Wheel catch the ball in flight? As students watch the two sets of equations being graphed simultaneously, they can use the ENTER button to pause at any point and simply press ENTER again to resume graphing.

Students will quickly find that getting the ball to the person on the Ferris Wheel with a close enough distance for the person to catch the ball is not likely in this case, as the series of screen captures in Figure 3 shows.

In a more detailed analysis, the Distance Formula can be used to measure the distance between the ball and the person riding the Ferris Wheel at any time $t$. Replacing the $x$- and $y$-coordinates of the familiar formula between two points $(x_1, y_1)$ and $(x_2, y_2)$ with the $x$- and $y$-components of the equations for the ball and the Ferris Wheel respectively, the distance between the ball and the person on the Ferris Wheel can be found by using the equation in Figure 4.

Now the question becomes whether or not this distance is ever zero or some small reasonable value, say less than or equal to 2, assuming that the person on the Ferris Wheel can reach out and catch the ball within 2 feet of his or her position. To test this, the new
Distance Formula can be stored as the function (see Figure 5) and analyzed with its graph and table. Figure 6 shows that about $t = 3.26$ seconds, the person on the Ferris Wheel is as close as he or she is going to get to the ball at a distance away from the ball of approximately 18.08 feet, making it impossible for the person to catch the ball.

Once it is determined that the original set-up will not “work,” I encourage students to further experiment by changing values in the parametric equations that will enable the person to be close enough to the ball to catch it. This makes the problem open-ended in nature, with the possibility of many solutions. Also, this causes students to pay attention to what the values in the equations mean in terms of physics context and develop rules about what they can change or not change in the equations. For example, considering the ball in flight, students can reasonably change the initial height of the ball being thrown, the initial velocity of the ball, and the angle at which the ball is thrown, but not the negative 16 that is associated with a constant acceleration due to gravity. Thus, the students can experiment with the model of the ball in flight

$$\begin{align*}
x &= 62 \cos (68.75) t \\
y &= 5 + 62 \sin (68.75) t - 16t^2
\end{align*}$$

by changing the 5 (initial height of the ball in feet), the 62 (initial velocity in ft/sec) in both the $x$- and $y$-components, and/or the 68.75 (angle in degrees) in both components.

Students could also consider whether or not to change the dimensions related to the Ferris Wheel, such as its center, radius, and/or rotation speed in the original model

$$\begin{align*}
x &= 20 \cos (40.11 t) + 100 \\
y &= 20 \sin (40.11 t) + 20.
\end{align*}$$

The center is affected by constant added to the $x$-component (100 feet) and the constant added to the $y$-component (20 feet). The radius and rotational speed are affected by the coefficient (20 feet) and angular velocity (40.11° per second), respectively, in the trigonometric functions in the $x$- and $y$-components (i.e., $20 \cos (40.11t)$ and $20 \sin (40.11t)$).
in the model). However, because it would not be easy for students to change the Ferris Wheel in real-life, I recommend against redesigning the wheel.

**A Sample Solution**

One of many possible solutions (briefly presented in Safier, 2004) involves keeping the original Ferris Wheel model as is, but adjusting the model for the ball in flight. By keeping the initial height at 5 ft, while adjusting the initial velocity to 72 ft/sec and the angle to 73.91°, sample solution models now involve a ball in flight in time \( t \) seconds as modeled by

\[
\begin{align*}
    x &= 72 \cos (73.91) t \\
    y &= 5 + 72 \sin (73.91) t - 16t^2
\end{align*}
\]

and a person on a Ferris Wheel, with this person's Ferris Wheel ride modeled by

\[
\begin{align*}
    x &= 20 \cos (40.11 t) + 100 \\
    y &= 20 \sin (40.11 t) + 20.
\end{align*}
\]

In this situation, the person on the Ferris Wheel can catch the ball about 4 seconds after it is thrown if it is assumed that the person can reach out and catch the ball within 2 feet of his or her position on the Ferris Wheel. This solution occurs when the person on the Ferris Wheel is near the 9 o’clock position at position (81.15, 26.70) while the ball is at nearby position (79.82, 25.72) leaving a distance of

\[
d = \sqrt{(79.82 - 81.15)^2 + (25.72 - 26.70)^2} \approx 1.65
\]

feet between the person and the ball. The graph shown in Figure 7 is paused to show the described solution along with related table values.

I have found it helpful to provide a calculator program of the Distance Formula to students during their experimental work on finding a solution to this problem. Having students work in groups of three allows for one student to scroll through position values of the ball in flight \((X_{1T}, Y_{1T})\), another student to scroll through position values of the person on the Ferris Wheel \((X_{2T}, Y_{2T})\) to compare and look for points close to the ball’s position \((X_{1T}, Y_{1T})\), and the third student to use the Distance Formula program to test distances between the ball in flight and the person on the Ferris Wheel. Figure 8 shows the use of the program for checking the solution in Figure 7.
Once again, treating the Distance Formula as a function serves as another method for verifying a solution. Given the new model of the ball in flight, the Distance Formula function becomes what is noted in Figure 9 with a minimum distance of 1.64 feet at $t \approx 4$ seconds. This is further depicted in Figure 10.

**Summary**

The possibilities of multiple correct solutions involving changes to
the original models for the problem and the capabilities of the TI-83/84 graphing calculator allow for open-ended exploration. I have found that students find this problem fun and rewarding in terms of the knowledge they gain about parametric equations. They enjoy taking on the challenge of getting the ball to the person on the Ferris Wheel and then comparing their analysis and results with other classmates’ work and solutions.

References


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