THE 2017 AMTNJ 27TH ANNUAL CONFERENCE

GROWTH MINDSTS IN MATHEMATICS AND IMPLEMENTING THE NEW

NEW JERSEY MATHEMATICS STANDARDS

OCTOBER 26-27, 2017

THE NATIONAL CONFERENCE CENTER AND THE HOLIDAY INN HOTEL

EAST WINDSOR, NJ

DATE AND TIME OF THE PRESENTATION: THURSDAY, OCTOBER 26, 2017

9:30 A.M. – 11:00 A.M.

SESSION NUMBER: 14

LOCATION: CONFERENCE ROOM D

TITLE OF PRESENTATION: PROBLEM SOLVING ACTIVITIES ARTICULATING THE PRACTICE OF CONSTRUCTING VIABLE ARGUMENTS

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PROBLEM SOLVING ACTIVITIES

ARTICULATING THE PRACTICE OF CONSTRUCTING VIABLE ARGUMENTS

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ABSTACT: Rich problem solving activities serve to articulate the eight Standards For Mathematical Practice alluded to in The Common Core document. This hands-on workshop will encompass many of these practices including the construction of viable arguments. Problems are selected from number, algebra, geometry, pre-calculus, calculus and discrete mathematics.

SOME PROBLEMS AND DISCUSSION ACTIVITIES:

I. Determine what occurs when one adds and multiplies the various combinations of even and odd integers. Form conjectures and create a table. Then try to justify formally.

II. Geometry and the Fibonacci sequence.
Consider any four consecutive terms in the Fibonacci sequence. First form the product of the first and fourth terms. Take twice the product of the second and third terms. Finally take the sum of the squares of the second and third terms in your sequence. Try to relate this to a theorem in plane geometry, conjecture based on several examples, and try to substantiate your conjecture.

III. Consider a rectangle with perimeters 20 and 32 units respectively. Determine the areas of all such rectangles having integer lengths and secure the dimensions of the rectangles with the largest possible areas. What do you conclude? Provide the necessary constraints on the dimensions of the rectangles. Can you generalize?

IV. (A Modeling with Mathematics Activity): At the beginning of the semester, each of the students becomes acquainted with one another by shaking hands. If the class consists of twenty-five students and each student shakes hands with everyone else apart from themselves, how many total handshakes are there? Try to approach your solution in each of the following ways:
a. Form a table and discern a pattern.
b. Try to formulate a rule establishing a correspondence between the number of people in the room and the number of handshakes and substantiate your rule.
c. Create a vertex-edge graph to model the situation.

V. Use both inductive reasoning (five cases) and then deductive reasoning to solve the following number puzzle employing the given directives:
a. Pick any number.
b. Add 221 to the given selected number.
c. Multiply the sum by 2652.
d. Subtract 1326 from your product.
d. Divide your difference by 663.
e. Subtract 870 from your quotient.
f. Divide your difference by 4.
g. Subtract the original number from your quotient.

VI. A Fun Activity with the Fibonacci sequence.

Consider the sum of any six consecutive terms in the Fibonacci sequence. Form the sum and divide by four. Try this with three different numerical data sets. Form a conjecture. Can you prove your conjecture? Repeat this problem for the sum of ten consecutive terms in the Fibonacci sequence. Form the sum and divide by eleven.

Next consider the sum of any fourteen consecutive terms in the Fibonacci sequence. Form the sum and divide by twenty-nine.

VII. As a seventh problem, the students determined the possible next term in the following sequence: 1, 2, 3, …. Possible answers were 4, 5, 6. All three answers were correct! How is this possible?

VIII. Determine the initial prime in the following sequence if it exists:
9, 98, 987, 9876, 98765, 987654, 9876543, 98765432, 9876543219, 98765432198,….

SOLUTIONS TO PROBLEMS AND ACTIVITIES:

I. If E represents the set of even integers and O connotes the set of odd integers, then empirical evidence for the four possible combinations of addition scenarios: $E+E=E, E+O=O, O+E=O, and\ O+O=E$ and the four possible combinations of multiplication scenarios: $E \cdot E=E, E \cdot O=E, O \cdot E=E, and\ O \cdot O=O.$ See **FIGURES 1-7** respectively with the aid of The VOYAGE 200 CAS Calculator from Texas Instruments:

F1 | F2 Algebra | F3 Calc | F4 Other | F5 PrgmIO | F6 Clean Up

■ 4 + 6	10
■ 16 + 24	40
■ 118 + 216	334
■ 1258 + 3464	4722
■ 13400 + 21654	35054

13400+21654

MAIN RAD AUTO FUNC 5/99

FIGURE 1

F1 | F2 Algebra | F3 Calc | F4 Other | F5 PrgmIO | F6 Clean Up

■ 4 + 5	9
■ 16 + 25	41
■ 187 + 450	637
■ 2452 + 3645	6097
■ 54288 + 6397	60685

54288+6397

MAIN RAD AUTO FUNC 5/99

FIGURE 2

F1 | F2 Algebra | F3 Calc | F4 Other | F5 PrgmIO | F6 Clean Up

■ 9 + 8	17
■ 45 + 86	131
■ 377 + 288	665
■ 4569 + 3412	7981
■ 72313 + 48988	121301

72313+48988

MAIN RAD AUTO FUNC 5/99

FIGURE 3

F1 | F2 Algebra | F3 Calc | F4 Other | F5 PrgmIO | F6 Clean Up

■ 9 + 7	16
■ 47 + 99	146
■ 311 + 259	570
■ 4113 + 5651	9764
■ 31257 + 41155	72412

31257+41155

MAIN RAD AUTO FUNC 5/99

FIGURE 4

F1 | F2 Algebra | F3 Calc | F4 Other | F5 PrgmIO | F6 Clean Up

■ 8·8	64
■ 34·96	3264
■ 426·314	133764
■ 4568·2344	10707392
■ 11222·34578	388034316

11222*34578

MAIN RAD AUTO FUNC 5/99

FIGURE 5

F1 | F2 Algebra | F3 Calc | F4 Other | F5 PrgmIO | F6 Clean Up

■ 4·5	20
■ 32·85	2720
■ 436·719	313484
■ 3456·3889	13440384
■ 45678·32315	1476084570

45678*32315

MAIN RAD AUTO FUNC 5/99

FIGURE 6

F1 | F2 Algebra | F3 Calc | F4 Other | F5 PrgmIO | F6 Clean Up

■ 5·6	30
■ 37·42	1554
■ 845·368	310960
■ 4567·3212	14669204
■ 45679·32124	1467392196

45679*32124

MAIN RAD AUTO FUNC 5/99

FIGURE 7

F1 | F2 Algebra | F3 Calc | F4 Other | F5 PrgmIO | F6 Clean Up

■ 9·9	81
■ 91·79	7189
■ 355·427	151585
■ 4853·3789	18388017
■ 77777·23331	1814615187

77777*23331

MAIN RAD AUTO FUNC 5/99

FIGURE 8

Based on reasoning inductively to a general conclusion via the observations of five specific cases, one can conjecture the following displayed in **TABLES 1 and 2:**

$+$	E	O
E	E	O
O	O	E

TABLE 1

$\times$	E	O
E	E	E
O	E	O

TABLE 2

To furnish a formal proof, we appeal to the definitions of even and odd integers:
We will verify four of the cases leaving the others for the interested reader.

(i). To prove that $E+E=E,$ let $a,b\in E.$ Then

$\exists k,l\in Z \ni (s.t.)\, a=2\cdot k \quad (10)$

$and\; b=2\cdot l. \quad (11)$

Now $a+b=2\cdot k+2\cdot l=2\cdot(k+l).$ $k,l\in\mathbb{Z}\Rightarrow k+l\in\mathbb{Z}.$ Call $k+l=s.$ Hence $a+b=2\cdot s\Rightarrow a+b\in E.$ $\square$

(ii). To prove $O+E=O,$ let $a\in O\; and\; b\in E.$ One can secure

$k,l\in\mathbb{Z}\ni a=2\cdot k+1 \quad (12).$

$and\; b=2\cdot l. \qquad (13).$

Next note that $a+b=2\cdot k+2\cdot l+1=(2\cdot k+2\cdot l)+1=2\cdot(k+l)+1.$ Observe that $k,l\in\mathbb{Z}\Rightarrow k+l\in\mathbb{Z}.$ Denote $k+l=s.$ Hence $a+b=2\cdot s+1\Rightarrow a+b\in O.$ $\square$

(iii). To show that $E\times O=E,$ let $a\in E\; and\; b\in O.$ One can thus find

$k,l\in\mathbb{Z}\ni a=2\cdot k \quad (13).$

$and\; b=2\cdot l+1 \qquad (14).$

$a\cdot b=(2\cdot k)\cdot(2\cdot l+1)=2\cdot k\cdot 2\cdot l+2\cdot k\cdot 1=4\cdot k\cdot l+2\cdot k=2\cdot(2\cdot k\cdot l+k).$ Appealing to the facts that $k,l\in\mathbb{Z}\Rightarrow k+l\in\mathbb{Z}\; and\; k\cdot l\in\mathbb{Z}$, we obtain $2\cdot k\cdot l+k\in\mathbb{Z}$.(Recall that $2\in\mathbb{Z}$.) Call $2\cdot k\cdot l+k=u.$ Hence $a\cdot b=2\cdot u\Rightarrow a\cdot b\in E.$ $\square$

(iv). Finally to demonstrate that $O\times O=O,$ let $a,b\in O.$ Then by definition,

$\exists k,l\in\mathbb{Z}\ni a=2\cdot k+1 \quad (15).$

$and\; b=2\cdot l+1 \qquad (16).$

$a\cdot b=(2\cdot k+1)\cdot(2\cdot l+1)=4\cdot k\cdot l+2\cdot k+2\cdot l+1=(4\cdot k\cdot l+2\cdot k+2\cdot l)+1=2\cdot(2\cdot k\cdot l+k+l)+1.$

Employing the closure properties of $+\,and\times in\,\mathbb{Z}$, one is assured that $2\cdot k\cdot l+k+l\in\mathbb{Z}.$ Call $2\cdot k\cdot l+k+l=v.$ Thus $a\cdot b=2\cdot v+1\Rightarrow a\cdot b\in O.$ $\square$

II. Geometry and the Fibonacci Sequence.

In this activity, we next take any four consecutive Fibonacci numbers. Form the product of the first and fourth terms of the sequence. Next take twice the product of the second and terms. Finally take the sum of the squares of the second and third terms. Observe the relationship to the Pythagorean Theorem in plane geometry. We gather some empirical evidence via the following three examples:

Example 1: Consider the set of four consecutive Fibonacci numbers $\{3,5,8,13\}$. Observe the truth of the following with the aid of the VOYAGE 200. See **FIGURE 9:**

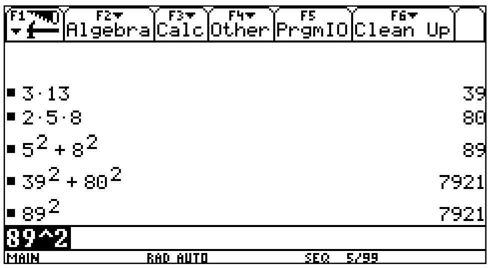

FIGURE 9

Observe that the primitive Pythagorean Triple (39, 80, 89) is formed.

Example 2: We next consider the sequence of four consecutive Fibonacci numbers $\{8,13,21,34\}$. We observe the truth of the following computations furnished by the VOYAGE 200. See **FIGURE 10:**

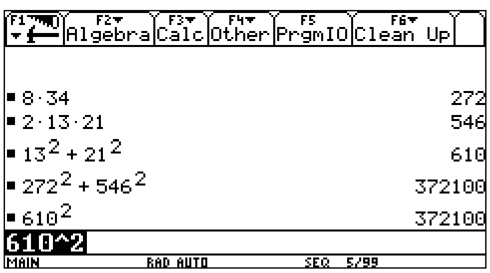

FIGURE 10

The Pythagorean Triple (272, 546, 610) (albeit not primitive; for 2 is a common factor among each of the components) is formed. The associated primitive Pythagorean Triple is (136, 273, 305).

Example 3: Consider the sequence of four consecutive Fibonacci numbers $\{13,21,34,55\}$. See **FIGURE 11** for the relevant computations.

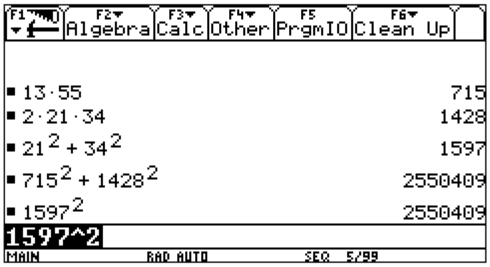

FIGURE 11

The primitive Pythagorean triple (715, 1428, 1597) is formed.
Note that the hypotenuses of each of the right triangles formed are Fibonacci numbers. (89, 610, 1597).

Based on the observations in the three examples, one suspects that a Pythagorean triple is always formed and this is indeed the case. We justify our conjecture with the aid of the VOYAGE 200:
Suppose $\{x, y, x+y, x+2\cdot y\}$ represent any four consecutive terms of the Fibonacci (or Fibonacci-like sequence). We view our inputs and outputs in **FIGURE 13** using the expand (command (See **FIGURE 12)** from the Algebra menu on the HOME SCREEN:

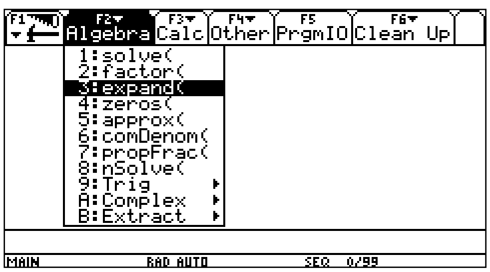

FIGURE 12

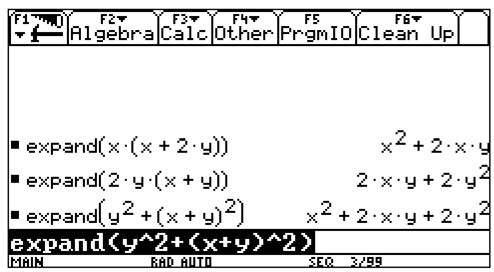

FIGURE 13

To show that $\left(x^2+2\cdot x\cdot y, 2\cdot x\cdot y+2\cdot y^2, x^2+2\cdot x\cdot y+2\cdot y^2\right)$ forms a Pythagorean Triple, see **FIGURES 14-16** for our inputs and outputs:

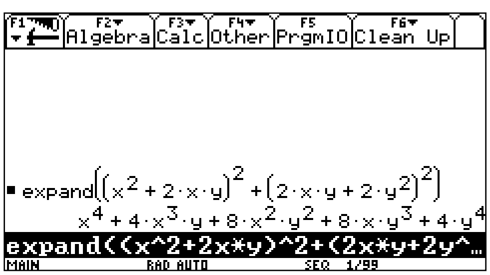

FIGURE 14

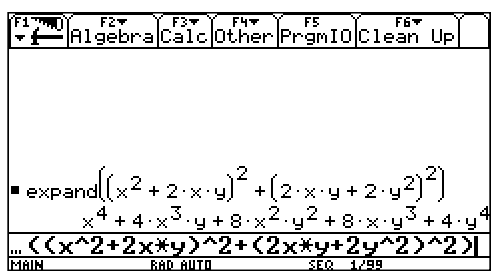

FIGURE 15

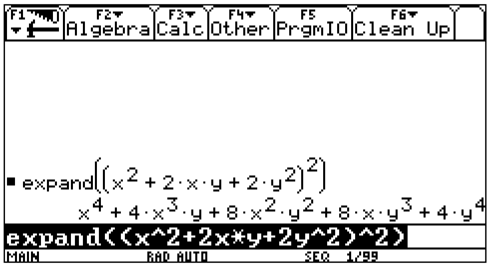

FIGURE 16

III. We know that the areas and perimeters of rectangles are given by the respective formulas $A = l \cdot w \; and \; P = 2 \cdot l + 2 \cdot w = 2 \cdot (l + w)$; $l = length \; and \; w = width.$ A student in the earlier grades might form tables such as the following for the respective perimeters of 20 and 100:

l:	w:	P:	A:
10	0	20	0
9	1	20	9
8	2	20	16
7	3	20	21
6	4	20	24
5	5	20	25
4	6	20	24
3	7	20	21
2	8	20	16
1	9	20	9
0	10	20	0

l:	w:	P:	A:
16	0	32	0
15	1	32	15
14	2	32	28
13	3	32	39
12	4	32	48
11	5	32	55
10	6	32	60
9	7	32	63
8	8	32	64
7	9	32	63
6	10	32	60
5	11	32	55
4	12	32	48
3	13	32	39
2	14	32	28
1	15	32	15
0	16	32	0

Based upon the outcomes in the table, it appears that the area of the largest rectangles having respective perimeters of 20 and 32 are the respective 5×5 *and* 8×8 squares. Is this always the case; namely that the area of largest rectangle having a given perimeter is necessarily a square whose length is one-quarter of the perimeter? Stay tuned. Progressing along, the student of algebra can graph the equations for the area utilizing algebra. For example, if

$$P = 20, \text{ then } 2 \cdot (l + w) = 20 \Rightarrow l + w = 10 \Rightarrow w = 10 - l \Rightarrow A = l \cdot w \Rightarrow A = l \cdot (10 - l) \Rightarrow A = 10 \cdot l - l^2.$$

Using the TI-84 or the TI-89, one can graph this equation and secure the maximum area (see **FIGURES 17-26):**

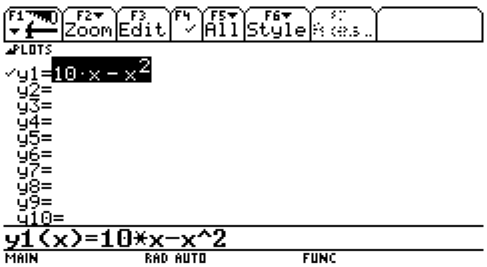

FIGURE 17

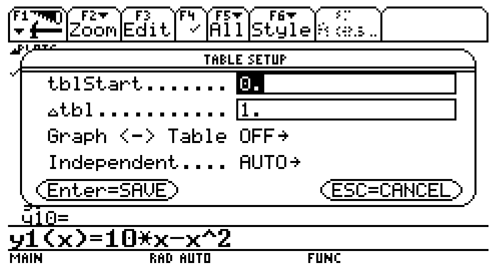

FIGURE 18

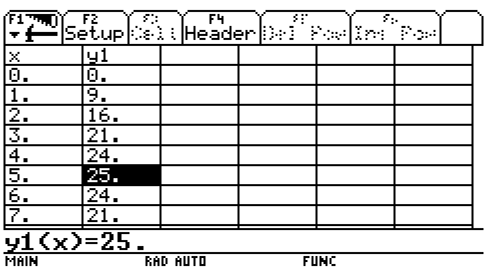

FIGURE 19

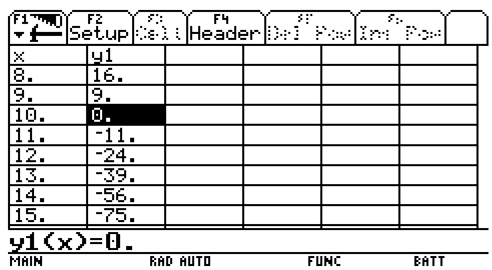

FIGURE 20

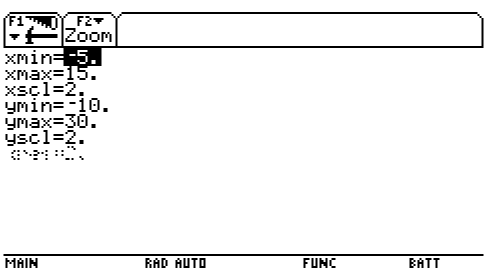

FIGURE 21

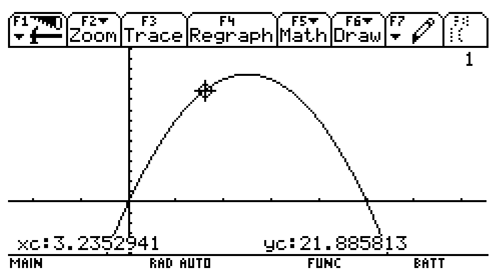

FIGURE 22

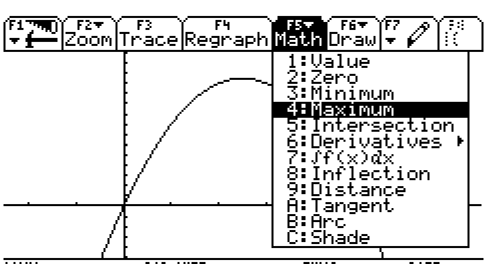

FIGURE 23

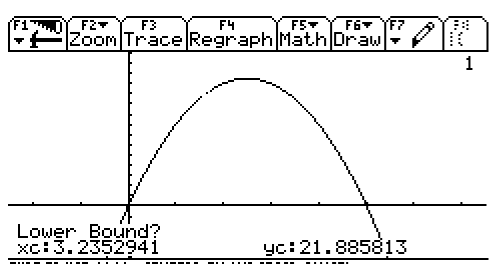

FIGURE 24

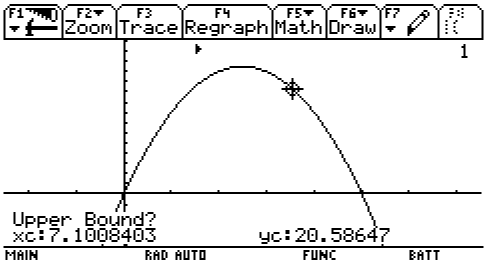

FIGURE 25

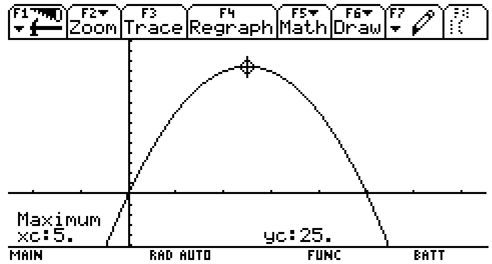

FIGURE 26

On the other hand, if

$P=32$, *then* $2\cdot(l+w)=32 \Rightarrow l+w=16 \Rightarrow w=16-l \Rightarrow A=l\cdot w \Rightarrow A=l\cdot(16-l) \Rightarrow A=16\cdot l-l^2.$

Using the TI-84 or the TI-89, one can graph this equation and secure the maximum area (see **FIGURES 27-37):**

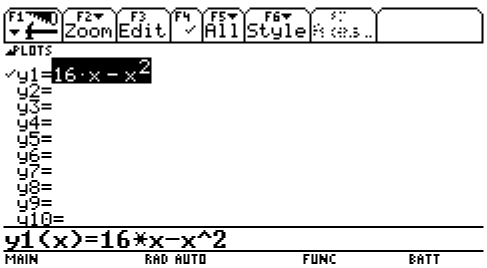

FIGURE 27

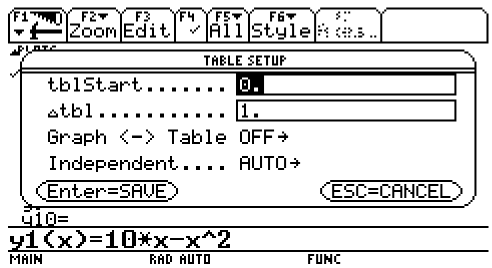

FIGURE 28

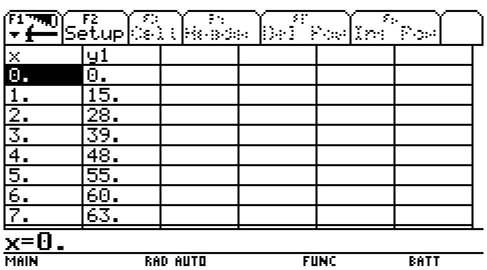

FIGURE 29

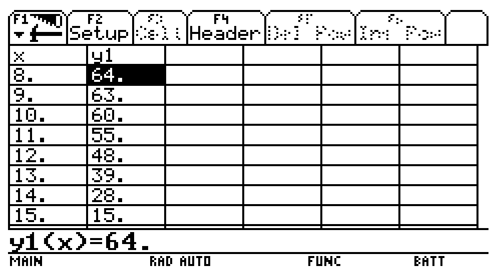

FIGURE 30

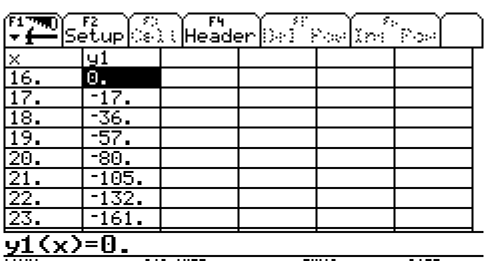

FIGURE 31

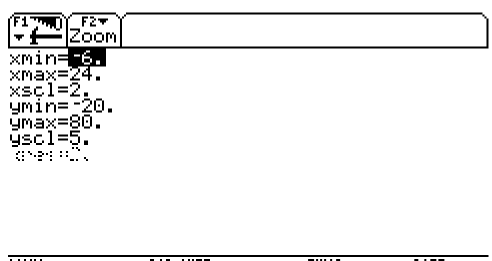

FIGURE 32

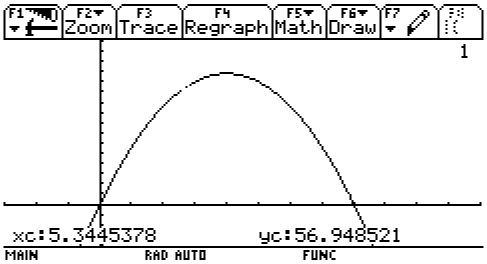

FIGURE 33

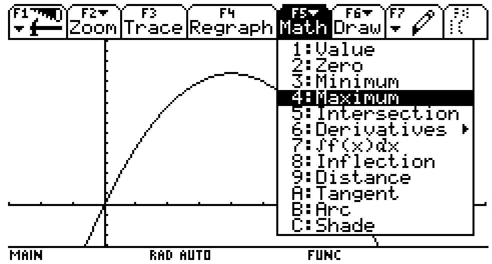

FIGURE 34

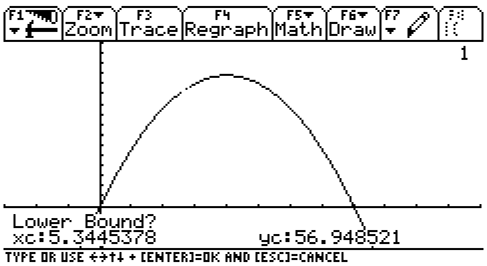

FIGURE 35

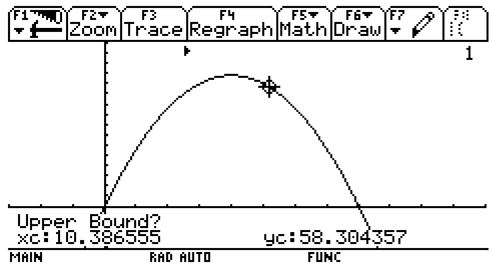

FIGURE 36

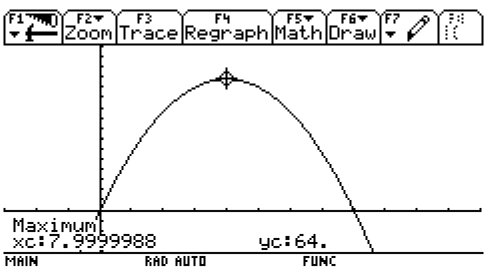

FIGURE 37

Observe that there are constraints on l *and* w; *namely that* $0 \le l \le \frac{P}{2}$ *and* $0 \le w \le \frac{P}{2}$; for otherwise the other dimension would be negative. Hence for rectangles having perimeters 20 and 32 respectively, the lengths and widths could not exceed 10 and 16 respectively. We still have not resolved the general question; namely what is the area of the largest rectangle having a given perimeter P? We prove using calculus that one always obtains a square each of whose sides has length $l = \frac{P}{4}$.

$A = l \cdot w\ (1)$ *and* $P = 2 \cdot l + 2 \cdot w.\ (2)$ The area formula represents the primary equation; for we are seeking to maximize this quantity and the perimeter formula which serves to aid us in our task is the secondary equation. Solving (2) for w and substituting into (1), we obtain

$$P = 2 \cdot l + 2 \cdot w \Rightarrow P - 2 \cdot l = 2 \cdot w \Rightarrow \frac{P - 2 \cdot l}{2} = \frac{P}{2} - l = w \Rightarrow A(l) = l \cdot \left(\frac{P}{2} - l\right) = \frac{P \cdot l}{2} - l^2. \ (3)$$

Differentiating (3) with respect to l and setting $\frac{dA}{dl} = 0$, we obtain

$\frac{dA}{dl} = \frac{P}{2} - 2 \cdot l$ *and* $\frac{dA}{dl} = 0 \Leftrightarrow \frac{P}{2} - 2 \cdot l = 0 \Leftrightarrow \frac{P}{2} = 2 \cdot l \Leftrightarrow \frac{P}{4} = l.$ $l = \frac{P}{4}$ represents the critical number of the area function $A(l)$. Using the second derivative test for relative extrema, we note that $\frac{d^2A}{dl^2} = -2 < 0 \Rightarrow l = \frac{P}{4}$ leads to a relative maximum of the area function $A(l)$. Substituting this value into (2), one obtains $A\left(\frac{P}{4}\right) = \left(\frac{P}{4}\right) \cdot \left(\frac{P}{2} - \frac{P}{4}\right) = \left(\frac{P}{4}\right) \cdot \left(\frac{P}{4}\right) = \frac{P^2}{16}$. To see that this is the largest possible area, note that since $A(l)$ is a polynomial function and hence continuous over $\mathbb{R}$ and hence over $\left[0, \frac{P}{2}\right] \subseteq \mathbb{R}$, the Extreme Value Theorem guarantees the existence of both an absolute maximum M and an absolute minimum m somewhere over $\left[0, \frac{P}{2}\right]$. We use the Tabular Method below to locate the absolute extrema where the largest of the tabulated values represents the absolute maximum of the area function over the interval and the smallest of the tabulated values represents the absolute minimum of the area function over the interval:

l:	$A(l) = \frac{P \cdot l}{2} - l^2$:
0	0
$\frac{P}{4}$	$\frac{P^2}{16}$
$\frac{P}{2}$	0

Note that $A\left(\frac{P}{2}\right) = \frac{P \cdot \left(\frac{P}{2}\right)}{2} - \left(\frac{P}{2}\right)^2 = \frac{\frac{P^2}{2}}{2} - \frac{P^2}{4} = \frac{P^2}{4} - \frac{P^2}{4} = 0.$ Hence the maximum area is obtained when one has a $\frac{P}{4} \times \frac{P}{4}$ square.

IV. To initiate our discussion, let us think of a simpler problem since it is rather elaborate as well as tedious to directly count the number of handshakes when there are 25 people in a room. If there is one person in the room, say A, clearly there are no handshakes since a person cannot shake hands with themselves. If there are two people in a room labeled A and B and they shake hands, this constitutes a single handshake, not two. (A shaking hands with B is identical to B shaking hands with A.) Next consider three people in a room denoted A, B, and C. Now C shakes hands with the two previous people called A and B. Hence two handshakes are added for a total of three. When a fourth person enters the room called D, this individual shakes hands with the previous three people A, B, and C so that three new handshakes are created for a total of six. One can continue in this manner and form the following table:

NUMBER OF PEOPLE IN THE ROOM	NUMBER OF HANDSHAKES
1	0
2	1
3	3
4	6
5	10
6	15
7	21
8	28
9	36
10	45
11	55
12	66
13	78
14	91
15	105
16	120
17	136
18	153
19	171
20	190
21	210
22	231
23	253
24	276
25	300

Hence we see that with 25 people in the room, one has a total of three hundred handshakes. One can perform the experiment with a smaller sample size and detect a pattern. In general if one has n people in a room and each shakes hands with all the other $n-1$ people save themselves, by **The Fundamental Principle of Counting,** one would expect there to be $n\cdot(n-1)$ handshakes. On the other hand, the order in which the handshakes occur is insignificant and this duality occurs for every pair of people. Hence we must divide the previous answer by two. Hence the total number of handshakes is $\frac{n\cdot(n-1)}{2}$. Hence if $n=25,$ one observes that $\frac{25\cdot(25-1)}{2}=\frac{25\cdot 24}{2}=25\cdot 12=300.$

Observe in the table that each time a new person is accounted for, the number of new handshakes increases by one over the previous time. In addition, if one multiplies any two consecutive entries in the left hand column and divides by two, they arrive at the entry in the right hand column corresponding to the second entry they selected from the left hand column. For example, $24\cdot 25=600 \, and \, \frac{600}{2}=300.$

A second way of viewing this problem is through the perspective of an entity known as an ***abstract graph*** where the **dots (vertices)** denote the people and the **lines (edges)** comprise the handshakes. Here one has a ***Complete Graph* on n vertices** K_n **,** in the sense that each vertex is connected to every vertex apart from itself via an edge. See the MATHEMATICA graphics on the last three pages for the Complete Graphs on 1, 2, 3, 4, 5, and 6 vertices respectively. One can do this until they reach K_{25}. Observe that the number of edges meeting at any vertex (called the *degree or valence* of the vertex) is always one less than the number of vertices in the graph K_n.

Another neat way of introducing vertex-edge graphs is to place paper plates on a classroom floor and connect the paper plates with paper tape. The vertices are the paper plates and the edges are the paper tape. This enables students to pictorially represent the degree or valence of a vertex (the number of edges incident on the vertex). An excellent text in which students solve rich problems on discrete mathematics is in the NCTM Navigations Series entitled ***Navigating Through Discrete Mathematics in Grades K-5 and Navigating Through Discrete Mathematics in Grades 6-12*** (a two volume set) written by Valerie DeBellis, Eric Hart, Margaret Kenney, and Joseph G. Rosenstein.

V. In inductive reasoning, we reason to a general conclusion via the observations of specific cases. The conclusions obtained via inductive reasoning are only probable but not absolutely certain. In contrast, deductive reasoning is the method of reasoning to a specific conclusion through the use of general observations. The conclusions obtained through the use of deductive reasoning are certain. In the following number puzzle, we employ the five specific numbers 5, 23, 12, 10, and 85 to illustrate inductive reasoning and then employ algebra to furnish a deductive proof. The puzzle and the solutions are provided below:

Pick any Number.	5	23	12	10	85	n
Add 221 to the given selected number.	226	244	233	231	306	n + 221
Multiply the sum by 2652.	599352	647088	617916	612612	811512	2652 n + 586092
Subtract 1326 from your product.	598026	645762	616590	611286	810186	2652 n + 584766
Divide your difference by 663.	902	974	930	922	1222	4 n + 882
Subtract 870 from your quotient.	32	104	60	52	352	4 n + 12
Divide your difference by 4.	8	26	15	13	88	n+3
Subtract the original number from your quotient.	3	3	3	3	3	3

The answer we obtain is always 3. We next deploy the calculator to show the inductive cases in **FIGURES 38-47** and the deductive case in **FIGURES 48-49:**

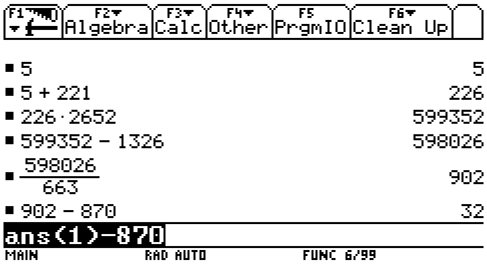

FIGURE 38

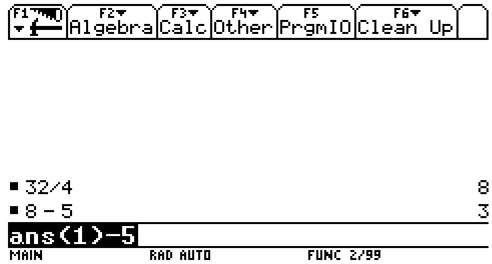

FIGURE 39

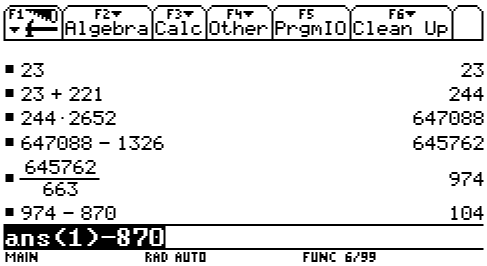

FIGURE 40

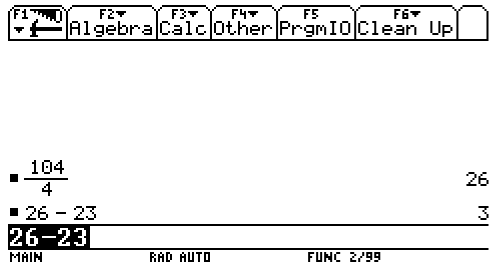

FIGURE 41

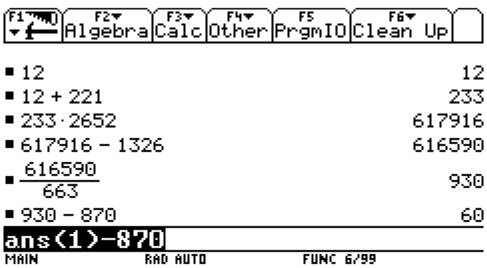

FIGURE 42

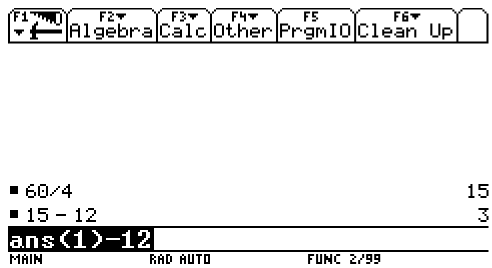

FIGURE 43

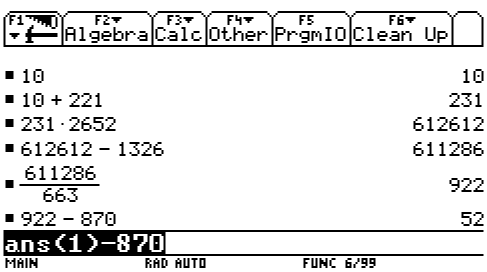

FIGURE 44

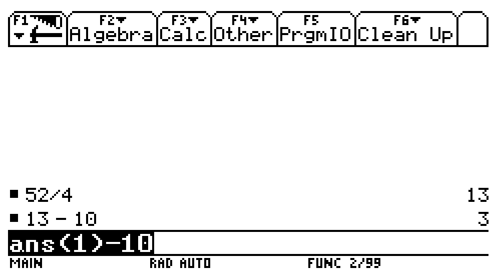

FIGURE 45

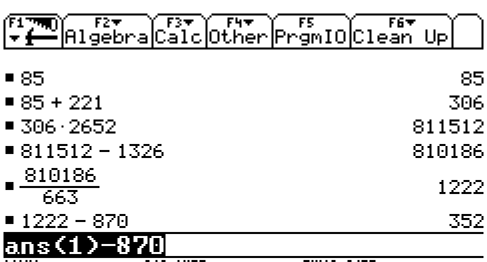

FIGURE 46

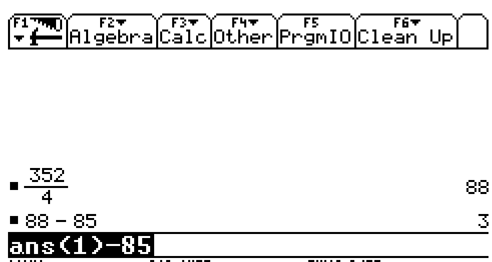

FIGURE 47

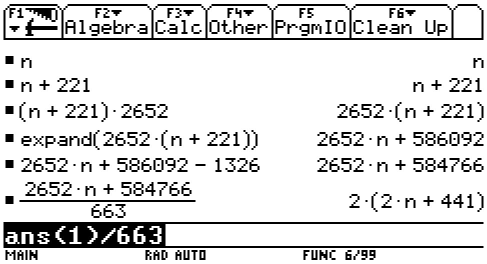

FIGURE 48

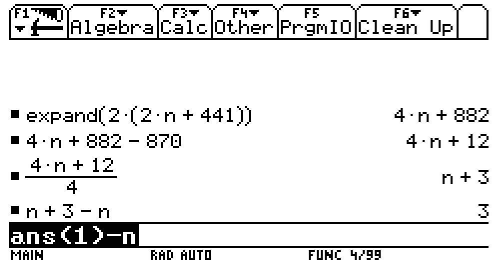

FIGURE 49

VI. A Fun Activity with the Fibonacci sequence.

If one considers the famous Fibonacci sequence or any Fibonacci-like sequence (that is a sequence whose first two terms can be anything one pleases but each term thereafter follows the recursion rule in the Fibonacci sequence), form the sum of any six consecutive terms and divide this sum by four. We do this for three separate sets and form a conjecture. The results are tabulated in the following TABLE:

SUM OF SIX CONSECUTIVE FIBONACCI NUMBERS:	SUM OF THE MEMBERS OF THE SET:	QUOTIENT WHEN THE SUM IS DIVIDED BY 4:
$\{2,3,5,8,13,21\}$	52	13…FIFTH TERM
$\{1,1,2,3,5,8\}$	20	5…FIFTH TERM
$\{55,89,144,233,377,610\}$	1508	377…FIFTH TERM

CONJECTURE: **The sum of any six consecutive Fibonacci numbers is divisible by 4 and the quotient will always be the fifth term in the sequence.**

Proof: Consider the initial two terms of the Fibonacci sequence to be x *and* y. The six consecutive terms of the sequence are as follows:

$\{x, y, x+y, x+2\cdot y, 2\cdot x+3\cdot y, 3\cdot x+5\cdot y\}$. We employ the VOYAGE 200 to form the sum and divide the resulting sum by 4. See **FIGURE 50:**

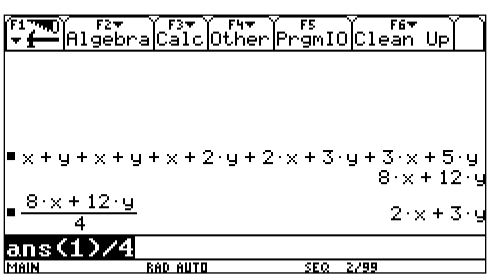

FIGURE 50

Let us next form the sum of any ten consecutive integers and divide this sum by 11. We do this for three separate sets and form a conjecture. The results are tabulated in the following TABLE:

SUM OF TEN CONSECUTIVE FIBONACCI NUMBERS:	SUM OF THE MEMBERS OF THE SET:	QUOTIENT WHEN THE SUM IS DIVIDED BY 11:
$\{2,3,5,8,13,21,34,55,89,144\}$	374	34…SEVENTH TERM
$\{1,1,2,3,5,8,13,21,34,55\}$	143	13…SEVENTH TERM
$\{55,89,144,233,377,610,987,1597,2584,4181\}$	10857	987…SEVENTH TERM

<u>CONJECTURE:</u> The sum of any ten consecutive Fibonacci numbers is divisible by 11 and the quotient will always be the seventh term in the sequence.

<u>Proof:</u> Consider the initial two terms of the Fibonacci sequence to be x *and* y. The ten consecutive terms of the sequence are as follows:

$$\{x, y, x+y, x+2\cdot y, 2\cdot x+3\cdot y, 3\cdot x+5\cdot y, 5\cdot x+8\cdot y, 8\cdot x+13\cdot y, 13\cdot x+21\cdot y, 21\cdot x+34\cdot y\}$$

Let us employ the TI-89 to form the sum and divide the resulting sum by 11. See **FIGURES 51-54:**

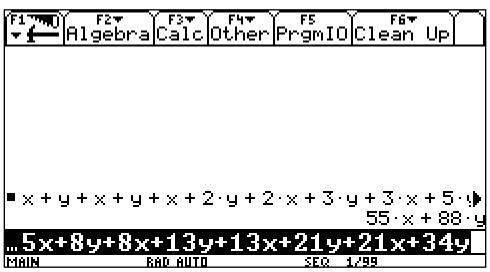

FIGURE 51

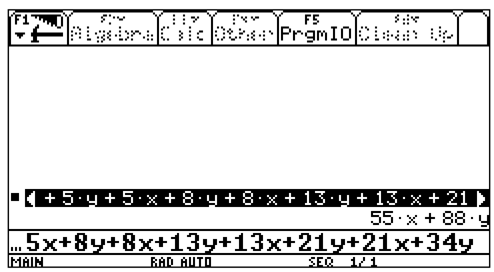

FIGURE 52

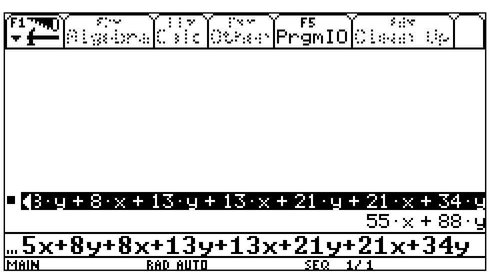

FIGURE 53

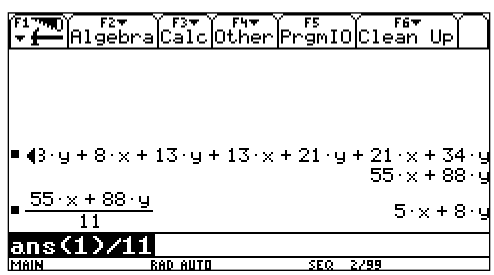

FIGURE 54

Notice $5\cdot x+8\cdot y$ is the seventh term in the sequence which is a neat Fibonacci number trick.

Let us next form the sum of any fourteen consecutive integers and divide this sum by 29 for three separate sets and form a conjecture. The results are tabulated in the following TABLE:

SUM OF FOURTEEN CONSECUTIVE FIBONACCI NUMBERS:	SUM OF THE MEMBERS OF THE SET:	QUOTIENT WHEN THE SUM IS DIVIDED BY 29:
$\{2,3,5,8,13,21,34,55,89,144,\ 233,\ 377,\ 610,\ 987\}$	2581	89…NINTH TERM
$\{1,1,2,3,5,8,13,21,34,55,\ 89,\ 144,\ 233,\ 377\}$	986	34…NINTH TERM
$\left\{\begin{matrix}55,89,144,233,377,610,987,1597,2584,\\ 4181,6765,\ 10946,\ 17711,\ 28657\end{matrix}\right\}$	74936	2584…NINTH TERM

<u>CONJECTURE:</u> The sum of any fourteen consecutive Fibonacci numbers is divisible by 29 and the quotient will always be the ninth term in the sequence.

<u>Proof:</u> Consider the initial two terms of the Fibonacci sequence to be $x \text{ and } y.$ The fourteen consecutive terms of the sequence are as follows:

$$\left\{\begin{matrix}x, y, x+y, x+2\cdot y, 2\cdot x+3\cdot y, 3\cdot x+5\cdot y, 5\cdot x+8\cdot y, 8\cdot x+13\cdot y, 13\cdot x+21\cdot y, 21\cdot x+34\cdot y,\\ 34\cdot x+55\cdot y,\ 55\cdot x+89\cdot y,\ 89\cdot x+144\cdot y,\ 144\cdot x+233\cdot y\end{matrix}\right\}$$

Let us employ the VOYAGE 200 to form the sum and divide the resulting sum by 29. See **FIGURES 55-59:**

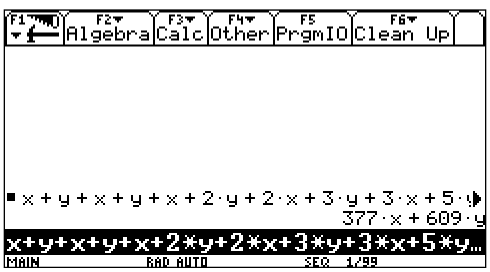

FIGURE 55

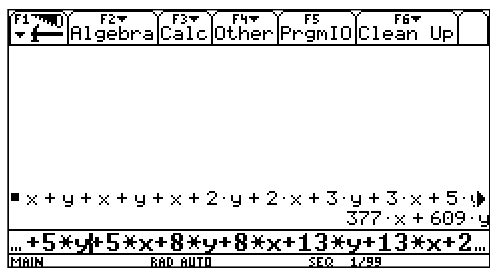

FIGURE 56

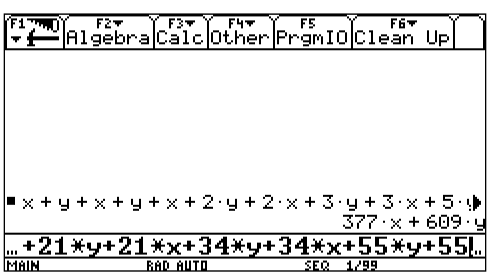

FIGURE 57

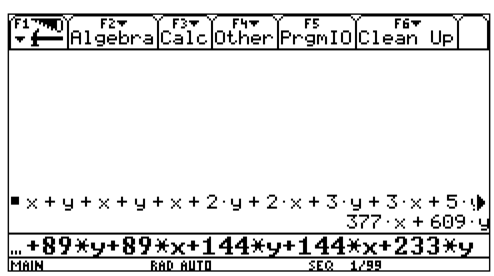

FIGURE 58

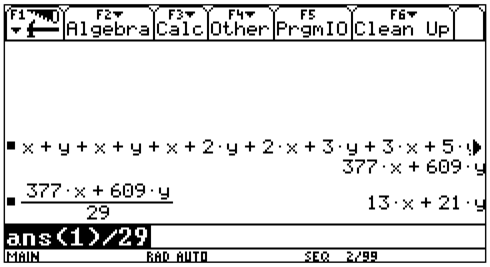

FIGURE 59

VII. All of the readers are correct; for giving a finite number of terms does not define a unique n^{th} term. An infinite number of n^{th} terms are indeed possible. In the first instance, one is looking at the arithmetic sequence 1, 2, 3, etc. to generate the next term 4. The next three terms are hence 5, 6 and 7. In the second instance, one is viewing a Fibonacci-like sequence in which the first two terms are 1 and 2 and each subsequent term is the sum of the previous two terms. The next three terms after 5 would hence be 8, 13 and 21 respectively. In the third instance, we view the first three terms 1, 2 and 3. Each term thereafter is the sum of the three immediate predecessors forming a Tribonacci-like sequence. Thus the fourth term is 6 and the next three subsequent terms are 11, 20 and 37.

VIII. We show that there are no primes in this sequence. Observe that the first term 9 is divisible by 3, the terms 98, 9876, 987654 and 98765432 are even and the term 98765 is divisible by 5. It is well known that any integer is divisible by 3 or 9 if and only if the integer obtained by forming the digital sum is divisible by 3 or 9. As a consequence, the terms 987654321 and 9876543219 are divisible by both 3 and 9. If one appends the digits 987, 9876, 987654, 9876543 and 987654321 to the integer 987654321 then divisibility by 3 is preserved. Clearly 98765432198765 is divisible by 5 and 98765432198765432 is divisible by 2. Hence there are no primes in this sequence. Of course, if one desired to view divisibility patterns, it would be helpful to factor the integers in question.

THANK YOU FOR YOUR PARTICIPATION DURING THE 27TH AMTNJ ANNUAL CONFERENCE IN EAST WINDSOR, NJ!