THE ASSOCIATION OF MATHEMATICS TEACHERS OF NEW JERSEY

2018 ANNUAL WINTER CONFERENCE

FOSTERING GROWTH MINDSETS IN EVERY MATH CLASSROOM

CREATING PRODUCTIVE LEARNING ENVIRONMENTS

WEDNESDAY, FEBRUARY 7, 2018

RAMADA PLAZA HOTEL AND CONFERENCE CENTER

MONROE TOWNSHIP, NJ

SESSION AND TIME: SESSION 1: 8:00 A.M. – 9:00 A.M.

LOCATION: PRINCETON

TITLE OF PRESENTATION: PATTERNS, PUZZLES AND MAGIC TO FOSTER ENGAGEMENT

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PATTERNS, PUZZLES AND MAGIC TO FOSTER ENGAGEMENT

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ABSTRACT: Mathematics needs to engage students and promote meaningful discourse. In this hands-on workshop, participants will focus on solving rich problems selected from number, algebra, geometry, precalculus and discrete mathematics yielding meaningful patterns that are both fun and surprising and culminate in aha moments. Please join us in our magical journey.

SOME PROBLEMS AND DISCUSSION ACTIVITIES:

I. A magic square is a configuration such that the sum of the elements in every row, every column and along both diagonals is the same. This common sum is referred to as the magic sum. For example, if one places each of the first nine counting integers in the following 3×3 **array, a magic square is obtained:**

4	**3**	**8**
9	**5**	**1**
2	**7**	**6**

One might ask how the integers were placed in the respective cells. Observe that $\sum_{i=1}^{9} i = 1+2+3+4+5+6+7+8+9=45$ **and 45 must be apportioned equally among the three rows which implies that the sum of the entries in each row, column and diagonal is** $\frac{45}{3}=15.$ **Our initial goal hence is to determine which number can be placed in the center square which encompasses four sums as the entry in the** $(2,2)$ **position must be an addend in four different sums that total 15; namely from the second row, second column and both diagonals. The only candidate is 5. Observe that** $15=1+5+9=2+5+8=3+5+7=4+5+6.$ **We next focus on the entries in the diagonal corners; for they occur in three different sums that total 15. The entry in the** $(1,1)$ **position is an addend in sums from the first row, first column and the main diagonal. Similarly the entry in the** $(3,3)$ **positon is an addend in sums from the third row, third column and the main diagonal. Likewise, the entry in the** $(1,3)$ **positon is an addend in sums from the first row, third column and the off diagonal while the entry in the** $(3,1)$ **positon is an addend in sums from the third row, first column and the off diagonal. The only possibilities for numbers in the diagonal corners are 2, 4, 6 and 8. Observe that**

$$15=2+4+9=2+5+8=2+6+7,\ 15=4+2+9=4+3+8=4+5+6,$$
$$15=6+1+8=6+2+7=6+4+5 \ \textit{and}\ 15=8+1+6=8+2+5=8+3+4.$$

Thus if 4 is placed in the $(1,1)$ **positon, then 6 must be placed in the** $(3,3)$ **position. Meanwhile if 8 is placed in the** $(1,3)$ **position, then 2 must be placed in the** $(3,1)$ **position. This leaves the integers 1, 3, 7 and 9 to be placed in the other cells that are not in the center cell or in the cells on the diagonal corners. These integers are addends in only two different sums. The entry in the** $(1,2)$ **position is an addend in sums from the first row and second column. The entry in the** $(2,1)$ **position is an addend in sums from the second row and first column. The entry in the** $(2,3)$ **position is an addend in sums from the second row and**

third column. Finally the entry in the $(3,2)$ position is an addend in sums from the third row and second column. We note that

$15=1+5+9=1+6+8,\ 15=3+4+8=3+5+7,\ 15=7+2+6=7+3+5\ \textit{and}\ 15=9+1+5=9+2+4.$

Hence if we place 3 in the $(1,2)$ position, then we must place 7 in the $(3,2)$ position. Finally if we place 9 in the $(2,1)$ position, we must place 1 in the $(2,3)$ position. This completes the magic square.

With the above information, do each of the following:

(1). Rotate the magic square above 90° clockwise and 90° counterclockwise. Also rotate the magic square above both 180° clockwise and 180° counterclockwise. Illustrate the magic squares that you obtain. Are these really different magic squares?

(2). In addition, add 10 to each entry in the original magic square. Generate some conclusions.

(3). Multiply each entry in the original magic square by 5 and generate some conclusions.

II. Consider the magic square below and fill in the missing entries so that we have a magic square consisting solely of primes and the number one:

7	**61**	**43**
73	x	**1**
y	z	w

Open Ended Problems: Suppose we add the same constant to every term in the magic square above after finding the values of the unknowns. Determine scenarios in which one obtains no prime outputs as well as some prime outputs (depending on the constant you choose to add to all of the terms in the magic square above. Is there ever a case where all prime outputs are obtained?

III. As our third problem, the students determined the possible next term in the following sequence: 1, 2, 3, …. Possible answers were 4, 5, 6. All three answers were correct! How is this possible?

IV. Use both inductive reasoning (five cases) and then deductive reasoning to solve the following number puzzle employing the given directives: a. Pick any number.
b. Add 221 to the given selected number.
c. Multiply the sum by 2652.
d. Subtract 1326 from your product.
d. Divide your difference by 663.
e. Subtract 870 from your quotient.
f. Divide your difference by 4.
g. Subtract the original number from your quotient.

V. Show that one has a magic square of order 3 whose entries are consecutive primes. The magic square is shown below:

1480028159	**1480028153**	**1480028201**
1480028213	**1480028171**	**1480028129**
1480028141	**1480028189**	**1480028183**

VI. A Fun Activity with the Fibonacci sequence.

Consider the sum of any six consecutive terms in the Fibonacci sequence. Form the sum and divide by four. Try this with three different numerical data sets. Form a conjecture. Can you prove your conjecture? Repeat this problem for the sum of ten consecutive terms in the Fibonacci sequence. Form the sum and divide by eleven. Next consider the sum of any fourteen consecutive terms in the Fibonacci sequence. Form the sum and divide by twenty-nine.

VII. Geometry and the Fibonacci sequence.

Consider any four consecutive terms in the Fibonacci sequence. First form the product of the first and fourth terms. Take twice the product of the second and third terms. Finally take the sum of the squares of the second and third terms in your sequence. Try to relate this to a theorem in plane geometry, conjecture based on several examples, and try to substantiate your conjecture.

SOLUTIONS TO PROBLEMS AND ACTIVITIES:

I. (1). If we rotate the original magic square

4	**3**	**8**
9	**5**	**1**
2	**7**	**6**

90° clockwise, we obtain the following magic square:

2	**9**	**4**
7	**5**	**3**
6	**1**	**8**

If we rotate the original magic square 90° counterclockwise, we obtain the magic square below:

8	**1**	**6**
3	**5**	**7**
4	**9**	**2**

If we rotate the original magic square 180° clockwise or counterclockwise, we obtain the following magic square:

6	**7**	**2**
1	**5**	**9**
8	**3**	**4**

These magic squares are not different from the original one as rotations preserve the magic sum.

(2). If we add 10 to each entry in the original magic square, we still obtain a magic square with magic sum 45:

14	**13**	**18**
19	**15**	**11**
12	**17**	**16**

(3). If we multiply each entry in the original magic square by 5, we likewise obtain a magic square whose magic sum is five times that of the original magic square and hence 75.

20	**15**	**40**
45	**25**	**5**
10	**35**	**30**

II. We consider the magic square below and fill in the missing entries so that we have a magic square consisting solely of primes and the number one:

7	**61**	**43**
73	x	**1**
y	z	w

Based on the entries in Row 1, we observe that the sum of the entries is $7+61+43=111.$ 111 constitutes the magic sum; for the sum of the entries in every row, column and along both diagonals must be identical. Hence along Column 1, observe that $7+73+y=111 \Leftrightarrow 80+y=111 \Leftrightarrow y=31.$ Hence along the off diagonal, we have $31+x+43=111 \Leftrightarrow x+74=111 \Leftrightarrow x=37.$ Along the main diagonal, we obtain $7+37+w=111 \Leftrightarrow 44+w=111 \Leftrightarrow w=67.$ Along the second column, we have $61+37+z=111 \Leftrightarrow 98+z=111 \Leftrightarrow z=13.$ Hence our completed magic square is as follows:

7	**61**	**43**
73	**37**	**1**
31	**13**	**67**

Note that the magic sum of 111 is three times the number in the center square (37) which is always true in any 3×3 magic square.

If one adds the same constant k to every entry in a magic square, a magic square is still obtained with the magic sum being $k\cdot n+m$ where m denotes the magic sum and n is the number of rows (and columns) in the square. Hence if one adds 10 to each entry in the magic square above, the magic sum would be $111+3\cdot10=111+30=141.$ Note that the entries in the resulting magic square are as follows:

17	**71**	**53**
83	**47**	**11**
41	**23**	**77**

One notes that by adding the constant 6 to every entry in the original magic square, they obtain a magic square with magic sum 129 with eight of the nine entries being prime (with the exception of 49):

13	**67**	**49**
79	**43**	**7**
37	**19**	**73**

Adding the constant 46 to every entry in the original magic square yields a magic square with magic sum 249 with seven of the nine entries being prime (the exceptions being 119 and 77):

53	**107**	**89**
119	**83**	**47**
77	**59**	**113**

Adding the constant 60 to every entry in the original magic square yields a magic square with magic sum 291 with six of the nine entries being prime (the exceptions being 121, 133 and 91):

67	**121**	**103**
133	**97**	**61**
91	**73**	**127**

Adding the constant 180 to every entry in the original magic square yields a magic square with magic sum 651 with five of the nine entries being prime (the exceptions being 187, 253, 217 and 247):

187	**241**	**223**
253	**217**	**181**
211	**193**	**247**

Adding the constant 186 to every entry in the original magic square yields a magic square with magic sum 669 with four of the nine entries being prime (the exceptions being 247, 259, 187, 217 and 253):

193	**247**	**229**
259	**223**	**187**
217	**199**	**253**

Adding the constant 516 to every entry in the original magic square yields a magic square with magic sum 1659 with three of the nine entries being prime (the exceptions being 523, 559, 589, 517, 529 and 583):

523	**577**	**559**
589	**553**	**517**
547	**529**	**583**

Adding the constant 1146 to every entry in the original magic square yields a magic square with magic sum 3549 with two of the nine entries being prime (the exceptions being 1207, 1189, 1219, 1183, 1147, 1177 and 1159):

1153	**1207**	**1189**
1219	**1183**	**1147**
1177	**1159**	**1213**

Adding the constant 1896 to every entry in the original magic square yields a magic square with magic sum 5809 with one of the nine entries being prime (the exceptions being 1903, 1957, 1939, 1969, 1897, 1927, 1909 and 1963):

1903	**1957**	**1939**
1969	**1933**	**1897**
1927	**1909**	**1963**

Finally, adding the constant 110 to each entry in the original magic square results in a magic square having magic sum 441 with none of the entries as primes:

117	**171**	**153**
183	**147**	**111**
141	**123**	**177**

With regards to securing a magic square obtained by adding the same constant to the original magic square and obtaining all nine entries as primes, I have yet to find any after conducting a MATHEMATICA search through one billion!

III. All of the readers are correct; for giving a finite number of terms does not define a unique n^{th} term. In the first instance, one is looking at the arithmetic sequence 1, 2, 3, etc. to generate the next term 4. The next three terms are hence 5, 6 and 7. In the second instance, one is viewing a Fibonacci-like sequence in which the first two terms are 1 and 2 and each subsequent term is the sum of the previous two terms. The next three terms after 5 would hence be 8, 13 and 21 respectively. In the third instance, we view the first three terms 1, 2 and 3. Each term thereafter is the sum of the three immediate predecessors forming a Tribonacci-like sequence. Thus the fourth term is 6 and the next three subsequent terms are 11, 20 and 37. There is another way to obtain 5 as the next term if the initial three terms are 1, 2 and 3 respectively. Consider the number of partitions of the natural numbers using only positive integer addends where order is not important. The following is obtained where $p(n)$ denotes the number of unordered partitions of n:

$p(1) = 1\ (1)$
$p(2) = 2\ (2,\ 1+1)$
$p(3) = 3\ (3,\ 1+2,\ 1+1+1)$
$p(4) = 5\ (4,\ 1+3,\ 2+2,\ 1+1+2,\ 1+1+1+1)$

Using the above, one can show that $p(5) = 7,\ p(6) = 11\ \mathit{and}\ p(7) = 15.$

Other possibilities (infinitely many) can occur.

IV. In inductive reasoning, we reason to a general conclusion via the observations of specific cases. The conclusions obtained via inductive reasoning are only probable but not absolutely certain. In contrast, deductive reasoning is the method of reasoning to a specific conclusion through the use of general observations. The conclusions obtained through the use of deductive reasoning are certain. In the following number puzzle, we employ the five specific numbers 5, 23, 12, 10, and 85 to illustrate inductive reasoning and then employ algebra to furnish a deductive proof. The puzzle and the solutions are provided below:

Pick any Number.	5	23	12	10	85	n
Add 221 to the given selected number.	226	244	233	231	306	$n+221$
Multiply the sum by 2652.	599352	647088	617916	612612	811512	$2652n+586092$
Subtract 1326 from your product.	598026	645762	616590	611286	810186	$2652n+584776$
Divide your difference by 663.	902	974	930	922	1222	$4n+882$
Subtract 870 from your quotient.	32	104	60	52	352	$4n+12$
Divide your difference by 4.	8	26	15	13	88	$n+3$
Subtract the original number from your quotient.	3	3	3	3	3	3

The answer we obtain is always 3. We next deploy the calculator to show the inductive cases in **FIGURES 1-10** and the deductive case in **FIGURES 11-12:**

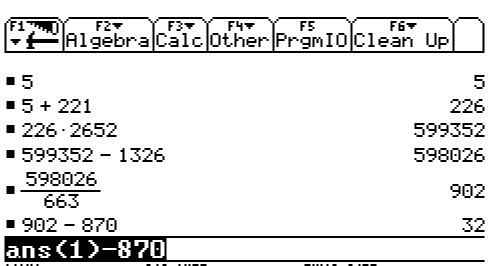

FIGURE 1

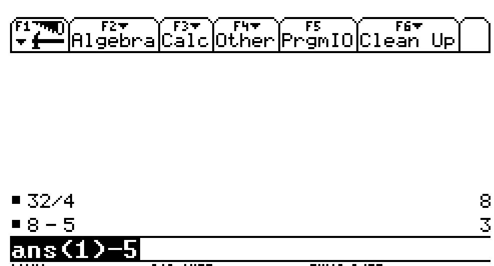

FIGURE 2

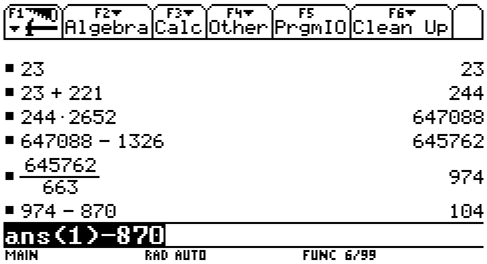

FIGURE 3

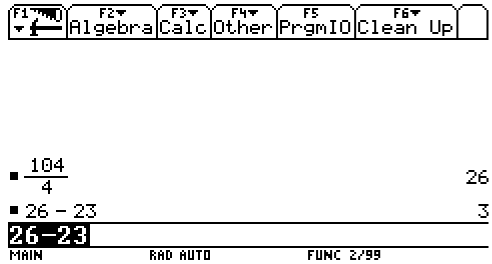

FIGURE 4

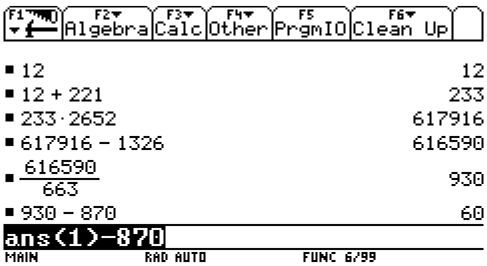

FIGURE 5

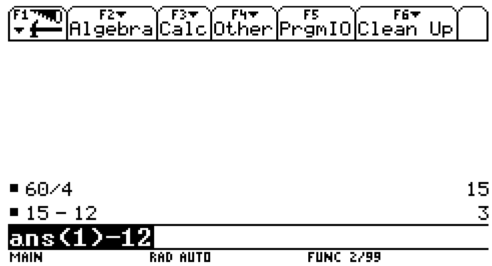

FIGURE 6

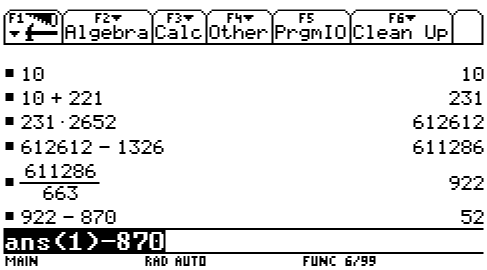

FIGURE 7

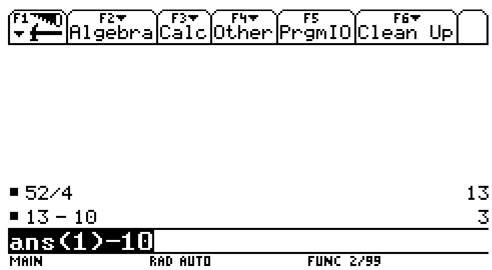

FIGURE 8

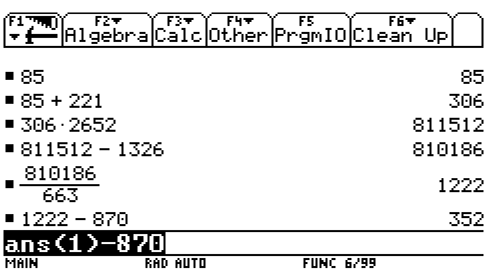

FIGURE 9

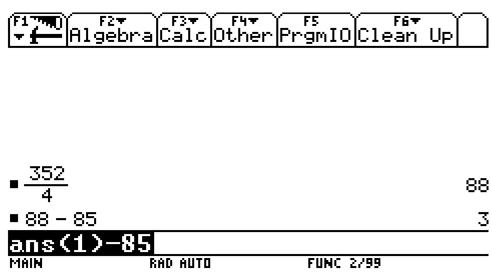

FIGURE 10

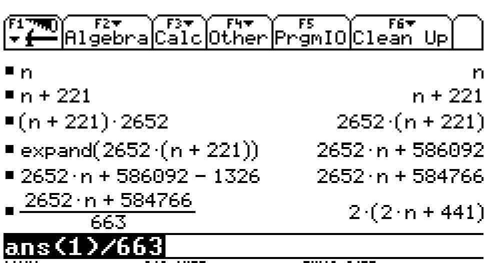

FIGURE 11

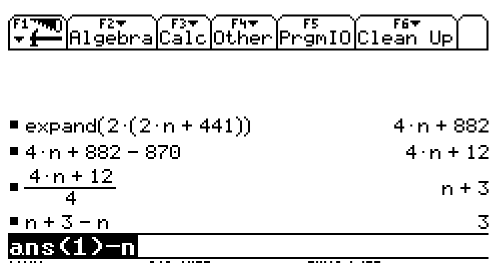

FIGURE 12

V. Recall that a **magic square is** a square such that the sum of the entries in every row, every column, and along both diagonals is always identical. This common sum is called the **magic constant.** Magic squares of size $n \times n$ always exist for $n \geq 3$. There is no 2×2 magic square. The magic constant (also called the **magic sum** for an $n \times n$ magic square) is given by the formula $\frac{n \cdot \left(n^2+1\right)}{2}$.

The following magic square of order 3 has nine entries each of which is a prime: (In addition, these primes are consecutive!)

1480028159	1480028153	1480028201
1480028213	1480028171	1480028129
1480028141	1480028189	1480028183

We first demonstrate that each of these integers is indeed prime in **FIGURES 13-14:**

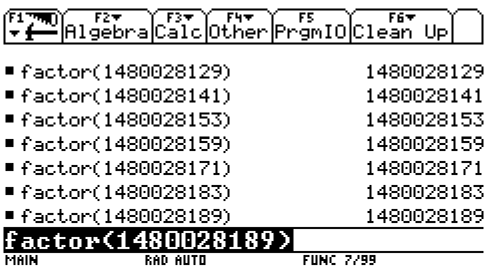

FIGURE 13

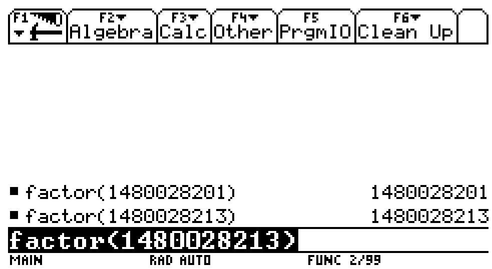

FIGURE 14

On Page 435 of the TI-89 manual, a program for the Next Prime is given. In **FIGURES 15-16,** we view the Program Ed (Program Editor) from the APPS MENU and **in FIGURE 17,** we view the program after pressing ENTER in **FIGURE 16:**

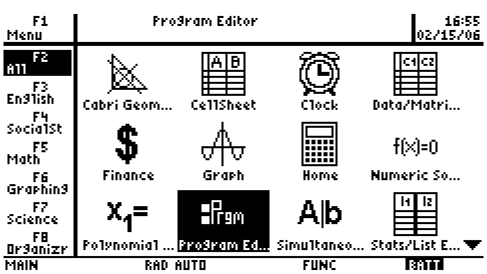

FIGURE 15

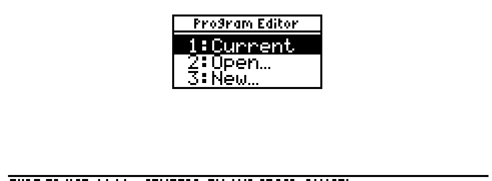

FIGURE 16

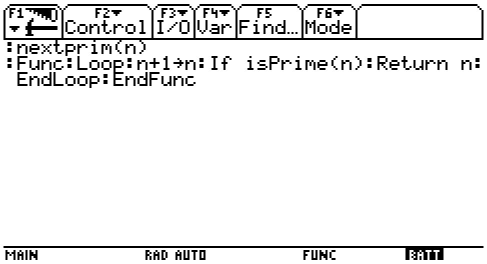

FIGURE 17

In **FIGURE 18,** we view the program in the Variables Link folder which is 2nd (-) (VAR LINK):

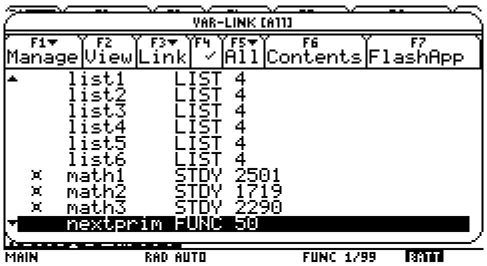

FIGURE 18

To cite a simple example, 11 is the prime successor to the prime 7 as we view in **FIGURE 19:**

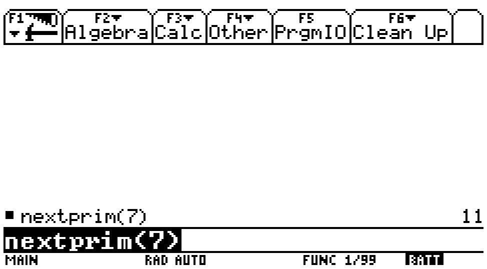

FIGURE 19

We next employ the Next Prime program to show that these nine primes are consecutive in **FIGURES 20-21:**

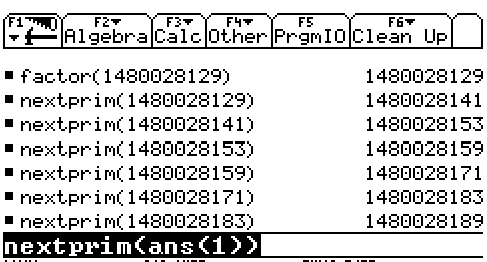

FIGURE 20

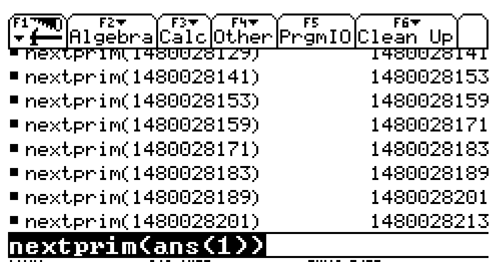

FIGURE 21

We next show that the configuration is indeed a magic square with the row sums in **FIGURES 22-23,** the column sums in **FIGURES 24-25** and the diagonal sums in **FIGURES 26-27** respectively:

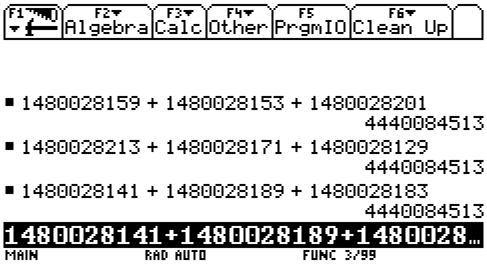

FIGURE 22

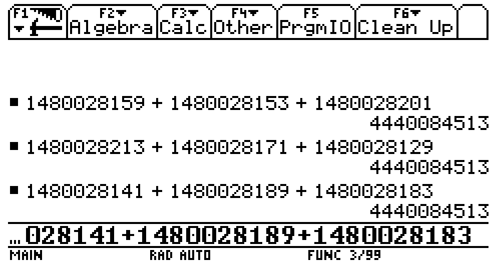

FIGURE 23

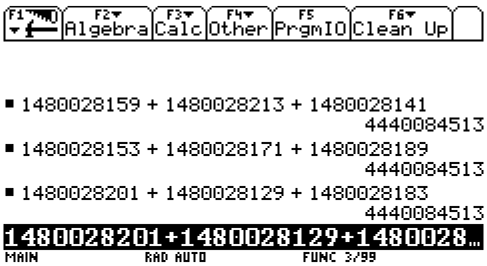

FIGURE 24

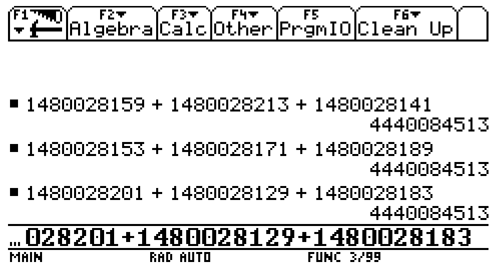

FIGURE 25

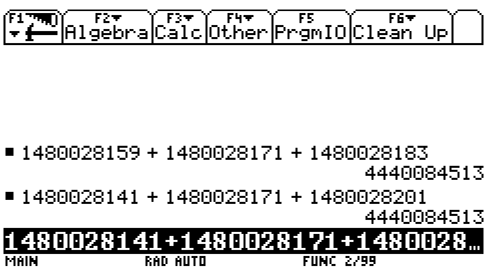

FIGURE 26

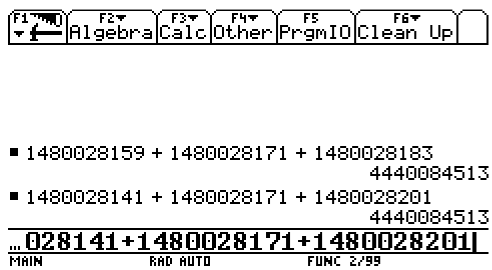

FIGURE 27

The magic sum of 4440084513 is not prime as seen in **FIGURE 28:**

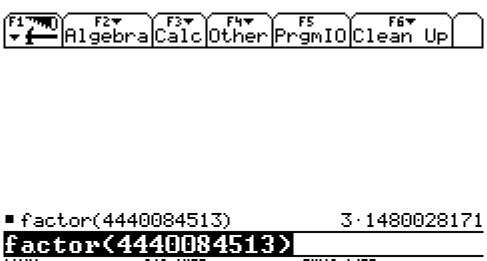

FIGURE 28

VI. A Fun Activity with the Fibonacci sequence

We generate the Fibonacci sequence on the HOME SCREEN. First recall the famous *Fibonacci sequence* is recursively defined as follows:

Define $F_1 = F_2 = 1$ *and* $F_n = F_{n-2} + F_{n-1}$ *for* $n \geq 3$. Here $F_n =$ the *nth* term of the Fibonacci sequence. We use the VOYAGE 200 to generate the initial forty outputs in the Fibonacci sequence. See **FIGURES 30-35:**

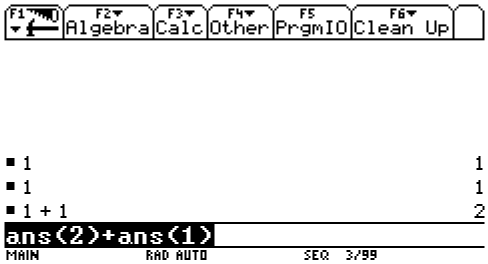

FIGURE 29

In **FIGURE 29,** on The HOME SCREEN, we entered the initial two terms to start the recursion which are both 1 and then used the command $ans(2)+ans(1)$ followed by ENTER. This will furnish the sum of the next to the last answer on the HOME SCREEN followed by the last answer on the HOME SCREEN. Keep pressing ENTER to generate new terms of this sequence. See **FIGURES 30-35:**

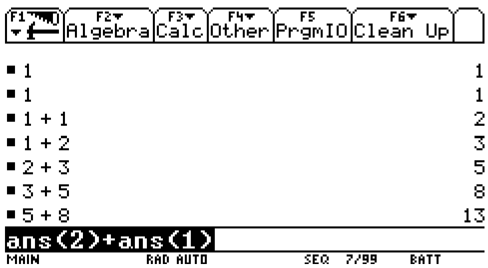

FIGURE 30

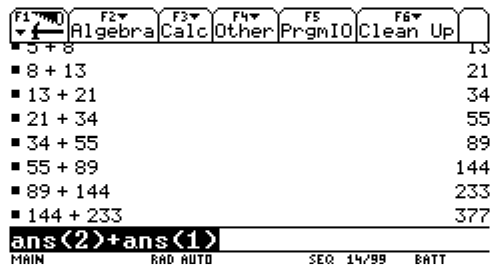

FIGURE 31

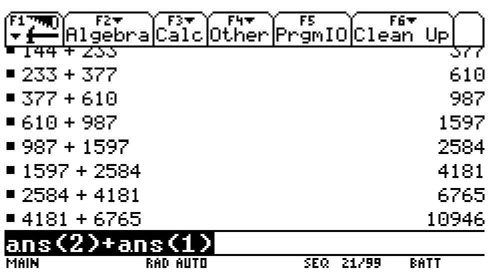

FIGURE 32

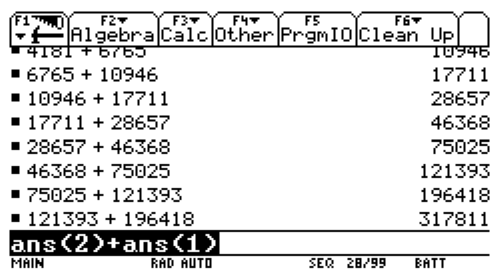

FIGURE 33

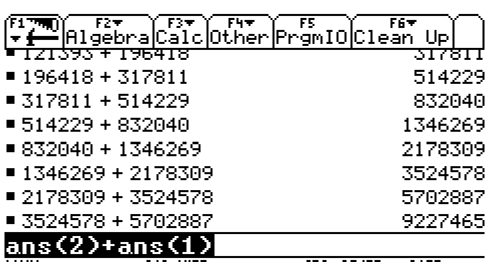

FIGURE 34

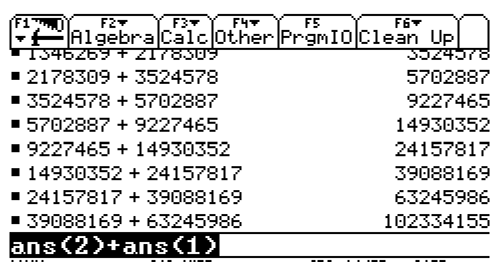

FIGURE 35

If one reads this data, they see two numbers on the bottom right; for example in **FIGURE 33,** one sees 28/99. The 28th term is the last answer in **FIGURE 33** and is 317811. There are 99 possible answers retained by the calculator. One can adjust this last number. From the HOME SCREEN, use the keystrokes F1 9: Format (see **FIGURE 36**) and press ENTER. You will see what is called History Pairs and use the right arrow cursor to see the choices, which indicate the number of answers one can recover from the HOME SCREEN (see **FIGURES 37-38**). The factory setting for the History Pairs is 30.

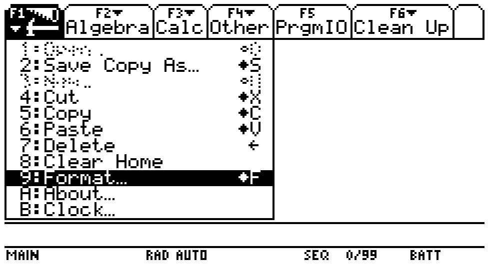

FIGURE 36

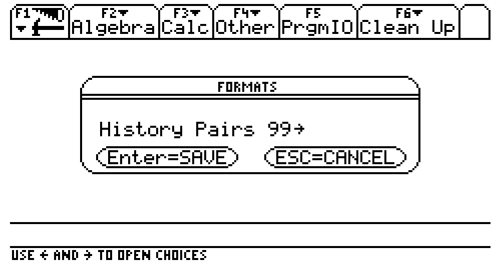

FIGURE 37

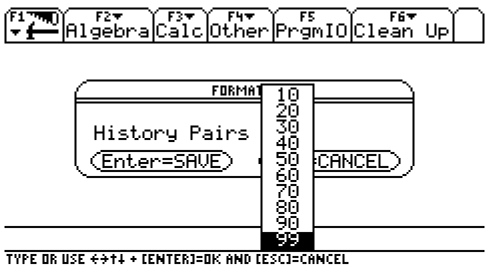

FIGURE 38

Based on the data in **FIGURES 30-35,** we conjecture that every fourth Fibonacci integer is divisible by three.

$$\left\{\begin{array}{l} F_4 = 3, F_8 = 21, F_{12} = 144, F_{16} = 987, F_{20} = 6765 \\ F_5 = 5, F_{10} = 55, F_{15} = 610, F_{20} = 6765, F_{25} = 75025 \\ F_8 = 21, F_{16} = 987, F_{24} = 46368, F_{32} = 2178309, F_{40} = 102334155 \end{array}\right\}.$$

Proceeding to SEQUENCE GRAPHING (use the keystrokes MODE followed by the right arrow cursor to option 4: SEQUENCE followed by ENTER), we see an SEQ at the bottom of the HOME SCREEN. See **FIGURES 39-40:**

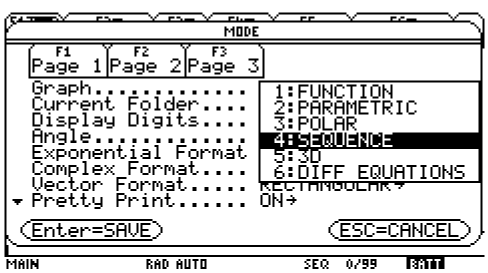

FIGURE 39

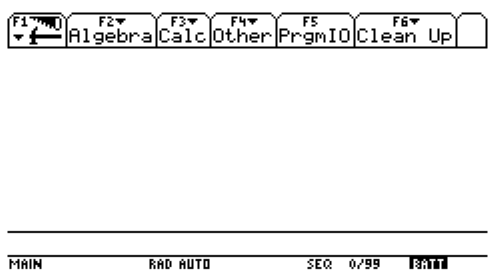

FIGURE 40

Next proceed to the Y= EDITOR and input the following as in **FIGURE 41** with the Standard Viewing Window, Graph, Table Setup, and a portion of the TABLE in **FIGURES 42-48:**

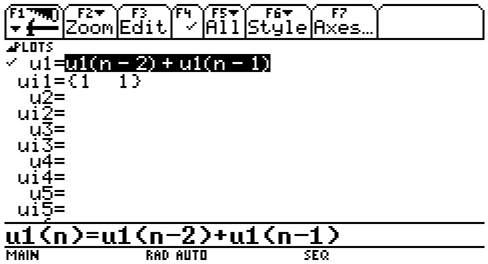

FIGURE 41

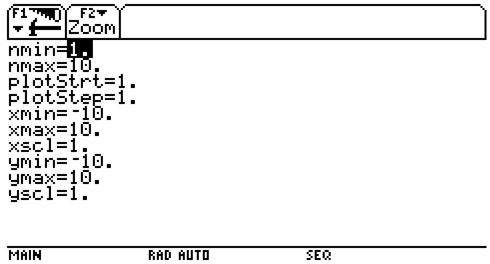

FIGURE 42

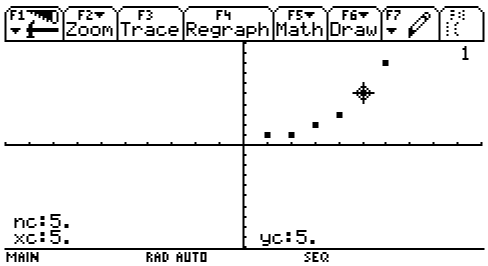

FIGURE 43

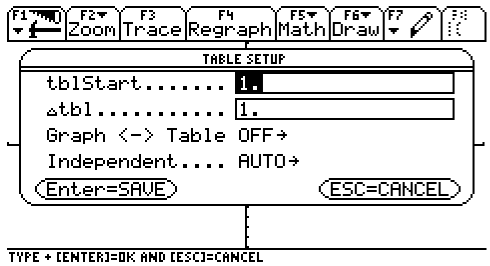

FIGURE 44

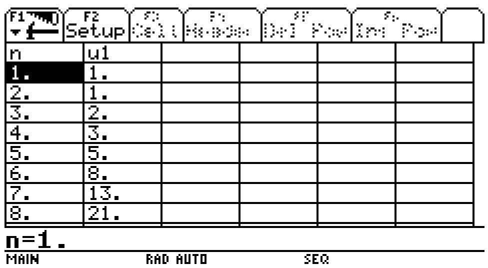

FIGURE 45

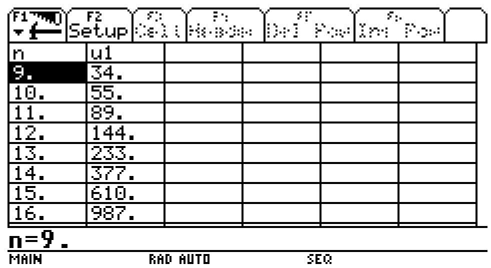

FIGURE 46

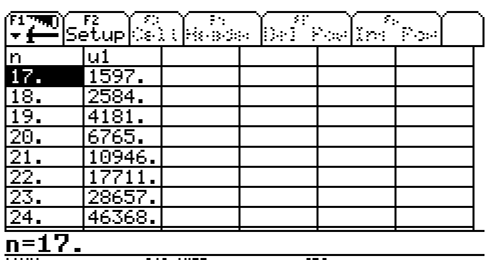

FIGURE 47

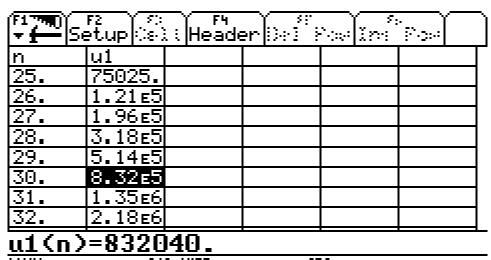

FIGURE 48

<u>Some comments on the above screen captures</u>:

1. **In FIGURE 41,** note that the recursion rule is provided on the line headed by $u1$ while the line headed by $ui1$ records the initial two terms of the sequence, the second followed by the first. There is no comma between the two 1's in Pretty Print although one separates the two initial 1's with a comma on the entry line. See **FIGURE 49:**

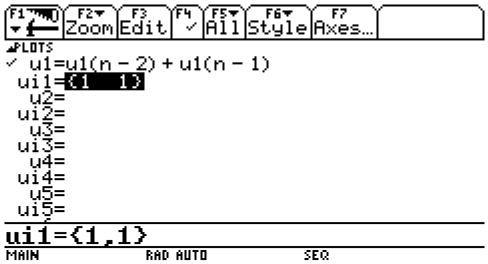

FIGURE 49

2. Note from **FIGURE 45** that 5 is the fifth term of the Fibonacci sequence.
3. Since a sequence is a function whose domain is the set of positive integers, the Tbl Start begins at 1 in **FIGURE 44.**
4. Only five figures are possible in any cell. Thus all terms of the Fibonacci sequence after the twenty-fifth are expressed in scientific notation. If one places their cursor on the output value, however, the exact value is determined as in **FIGURE 48** where the thirtieth term is given exactly as 832040.

Thus if one considers the famous Fibonacci sequence or any Fibonacci-like sequence (that is a sequence whose first two terms can be anything one pleases but each term thereafter follows the recursion rule in the Fibonacci sequence), form the sum of any six consecutive terms and divide this sum by four. We do this for three separate sets and form a conjecture. The results are tabulated in the following TABLE:

SUM OF SIX CONSECUTIVE FIBONACCI NUMBERS:	SUM OF THE MEMBERS OF THE SET:	QUOTIENT WHEN THE SUM IS DIVIDED BY 4:
$\{2,3,5,8,13,21\}$	52	13…FIFTH TERM
$\{1,1,2,3,5,8\}$	20	5…FIFTH TERM
$\{55,89,144,233,377,610\}$	1508	377…FIFTH TERM

CONJECTURE: The sum of any six consecutive Fibonacci numbers is divisible by 4 and the quotient will always be the fifth term in the sequence.

Proof: Consider the initial two terms of the Fibonacci sequence to be $x \text{ and } y$. The six consecutive terms of the sequence are as follows: $\{x, y, x+y, x+2\cdot y, 2\cdot x+3\cdot y, 3\cdot x+5\cdot y\}$. We employ the VOYAGE 200 to form the sum and divide the resulting sum by 4. See **FIGURE 49:**

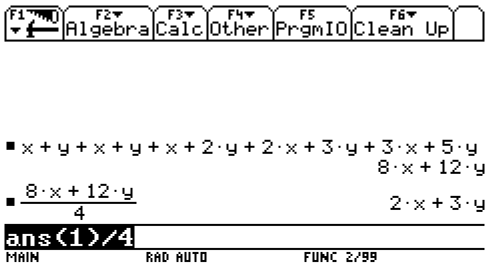

FIGURE 49

Let us next form the sum of any ten consecutive integers and divide this sum by 11. View this for three separate sets and form a conjecture. The results are tabulated below:

View this for three separate sets and form a conjecture. The results are tabulated below: SUM OF TEN CONSECUTIVE FIBONACCI NUMBERS:	SUM OF THE MEMBERS OF THE SET:	QUOTIENT WHEN THE SUM IS DIVIDED BY 11:
$\{2,3,5,8,13,21,34,55,89,144\}$	374	34…SEVENTH TERM
$\{1,1,2,3,5,8,13,21,34,55\}$	143	13…SEVENTH TERM
$\{55,89,144,233,377,610,987,1597,2584,4181\}$	10857	987…SEVENTH TERM

<u>CONJECTURE:</u> The sum of any ten consecutive Fibonacci numbers is divisible by 11 and the quotient will always be the seventh term in the sequence.

<u>Proof:</u> Consider the initial two terms of the Fibonacci sequence to be $x\ and\ y.$ The ten consecutive terms of the sequence are as follows:
$\{x, y, x+y, x+2\cdot y, 2\cdot x+3\cdot y, 3\cdot x+5\cdot y, 5\cdot x+8\cdot y, 8\cdot x+13\cdot y, 13\cdot x+21\cdot y, 21\cdot x+34\cdot y\}$. Let us employ the TI-89 to form the sum and divide the resulting sum by 11. See **FIGURES 50-53:**

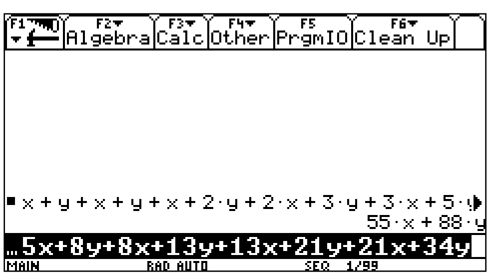

FIGURE 50

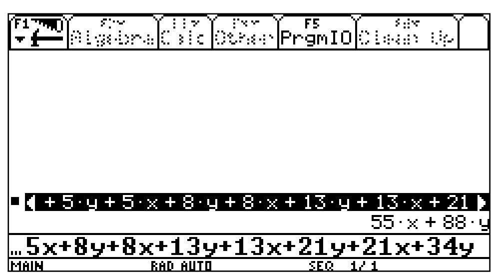

FIGURE 51

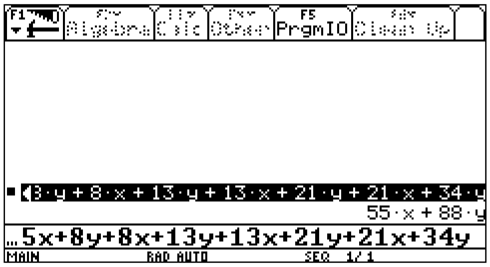

FIGURE 52

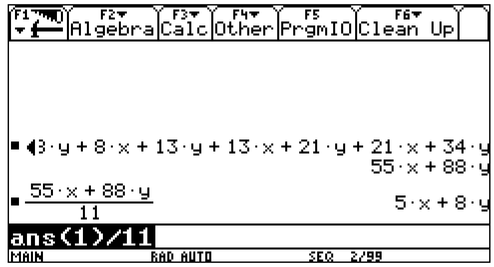

FIGURE 53

Notice $5\cdot x+8\cdot y$ is the seventh term in the sequence which is a neat Fibonacci number trick.

Let us next form the sum of any fourteen consecutive integers and divide this sum by 29 for three separate sets and form a conjecture. The results are tabulated below:

SUM OF FOURTEEN CONSECUTIVE FIBONACCI NUMBERS:	SUM OF THE MEMBERS OF THE SET:	QUOTIENT WHEN THE SUM IS DIVIDED BY 29:
$\{2,3,5,8,13,21,34,55,89,144,\ 233,\ 377,\ 610,\ 987\}$	2581	89…NINTH TERM
$\{1,1,2,3,5,8,13,21,34,55,\ 89,\ 144,\ 233,\ 377\}$	986	34…NINTH TERM
$\left\{\begin{matrix}55,89,144,233,377,610,987,1597,2584,\\ 4181,6765,\ 10946,\ 17711,\ 28657\end{matrix}\right\}$	74936	2584…NINTH TERM

<u>**CONJECTURE:**</u> **The sum of any fourteen consecutive Fibonacci numbers is divisible by 29 and the quotient will always be the ninth term in the sequence.**

<u>**Proof:**</u> Consider the initial two terms of the Fibonacci sequence to be $x \text{ and } y$. The fourteen consecutive terms of the sequence are as follows:

$$\left\{\begin{matrix}x,\ y,\ x+y,\ x+2\cdot y,\ 2\cdot x+3\cdot y,\ 3\cdot x+5\cdot y,\ 5\cdot x+8\cdot y,\ 8\cdot x+13\cdot y,\ 13\cdot x+21\cdot y,\ 21\cdot x+34\cdot y,\\ 34\cdot x+55\cdot y,\ 55\cdot x+89\cdot y,\ 89\cdot x+144\cdot y,\ 144\cdot x+233\cdot y\end{matrix}\right\}.$$

The VOYAGE 200 is used to form the sum and divide the total by 29. See **FIGURES 54-58:**

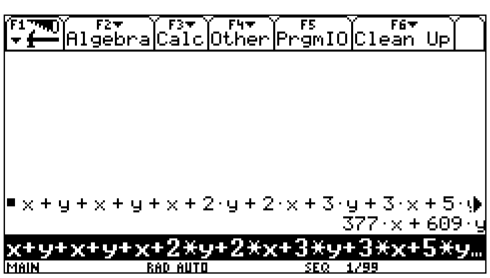

FIGURE 54

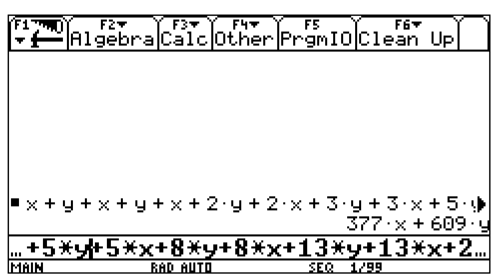

FIGURE 55

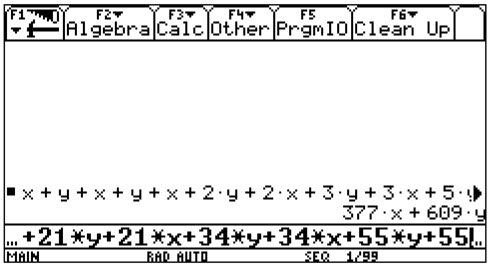

FIGURE 56

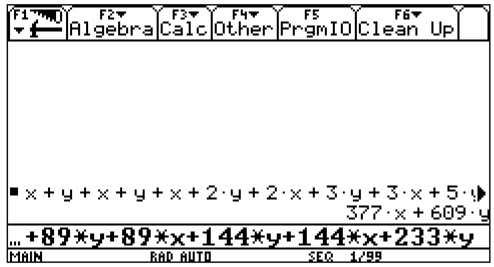

FIGURE 57

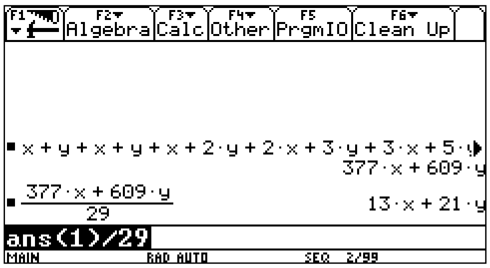

FIGURE 58

VII. <u>Geometry and the Fibonacci sequence.</u>

In this activity, we next take any four consecutive Fibonacci numbers. Form the product of the first and fourth terms of the sequence. Next take twice the product of the second and terms. Finally take the sum of the squares of the second and third terms. Observe the relationship to the Pythagorean Theorem in plane geometry. We gather some empirical evidence via the following three examples:

<u>Example 1:</u> Consider the set of four consecutive Fibonacci numbers $\{3,5,8,13\}$. Observe the truth of the following with the aid of the VOYAGE 200. See **FIGURE 59:**

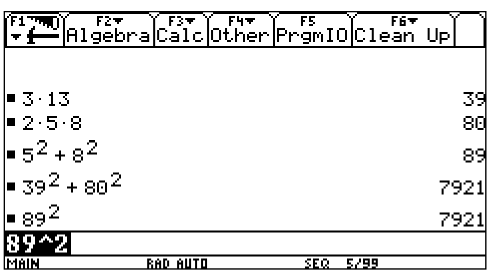

FIGURE 59

Observe that the primitive Pythagorean Triple (39, 80, 89) is formed.

<u>Example 2:</u> We next consider the sequence of four consecutive Fibonacci numbers $\{8,13,21,34\}$. We observe the truth of the following computations furnished by the VOYAGE 200. See **FIGURE 60:**

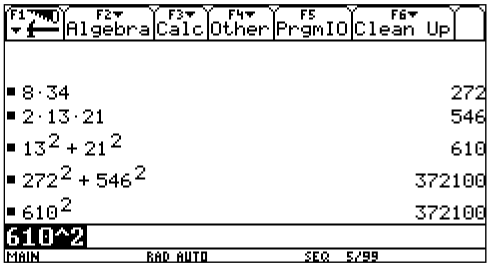

FIGURE 60

The Pythagorean Triple (272, 546, 610) (albeit not primitive; for 2 is a common factor among each of the components) is formed. The associated primitive Pythagorean Triple is (136, 273, 305).

Example 3: Consider the sequence of four consecutive Fibonacci numbers $\{13, 21, 34, 55\}$. See **FIGURE 61** for the relevant computations.

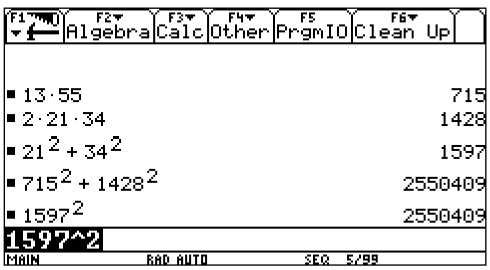

FIGURE 61

The primitive Pythagorean triple (715, 1428, 1597) is formed. Note that the hypotenuses of each of the right triangles formed are Fibonacci numbers. (89, 610, 1597).

Based on the observations in the three examples, one suspects that a Pythagorean triple is always formed and this is indeed the case. We justify our conjecture with the aid of the VOYAGE 200:

Suppose $\{x, y, x+y, x+2\cdot y\}$ represent any four consecutive terms of the Fibonacci (or Fibonacci-like sequence). We view our inputs and outputs in **FIGURE 63** using the expand (command (See **FIGURE 62)** from the Algebra menu on the HOME SCREEN:

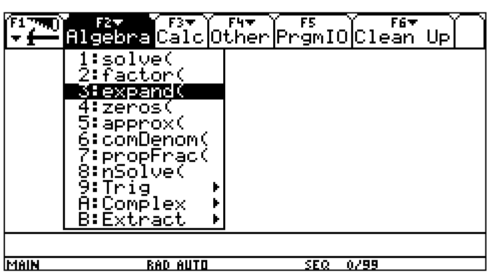

FIGURE 62

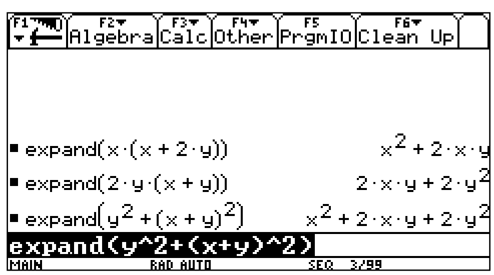

FIGURE 63

To show that $\left(x^2+2\cdot x\cdot y, 2\cdot x\cdot y+2\cdot y^2, x^2+2\cdot x\cdot y+2\cdot y^2\right)$ forms a Pythagorean Triple, see **FIGURES 64-66** for our inputs and outputs:

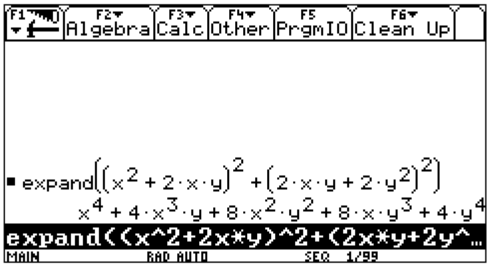

FIGURE 64

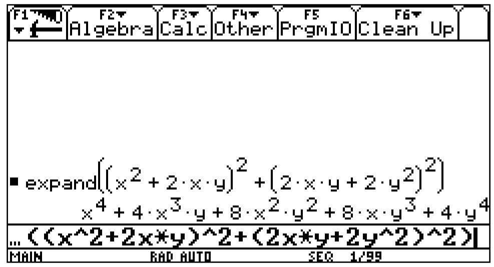

FIGURE 65

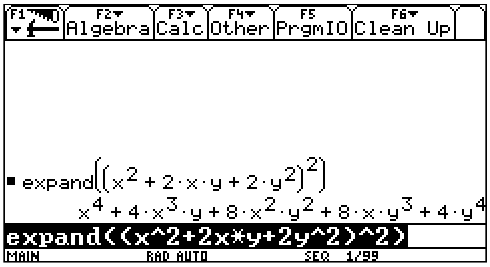

FIGURE 66

THANK YOU FOR YOUR PARTICIPATION AT THE 2018 ANNUAL WINTER CONFERENCE AT THE RAMADA PLAZA IN MONROE TOWNSHIP, NJ!