43rd Annual AMTNJ High School Mathematics Contest

Answer Key

1. $\frac{5}{6}$

2. 2500

3. 3

4. 36

5. $\frac{7}{12}$

6. $\frac{26}{127}$

7. 8

8. $\frac{7}{64}$ or 0.109

9. $2\sqrt{10}+\sqrt{74}\approx 14.927$

10. $\frac{39}{4}$ or 9.75 or $9\frac{3}{4}$

11. 16π or 50.265

12. $\sqrt{\frac{2\sqrt{3}}{\pi}}$ or 1.050

13. $\frac{1}{3}$

14. 9

15. 34

1. Find the exact value of $\dfrac{2^{2020}-2^{2016}}{2^{2020}+2^{2017}}$. Write your answer in simplest form.

Solution: $\dfrac{2^{2020}-2^{2016}}{2^{2020}+2^{2017}}=\dfrac{2^{2016}\left(2^4-1\right)}{2^{2016}\left(2^4+2\right)}=\dfrac{15}{18}=\dfrac{5}{6}$.

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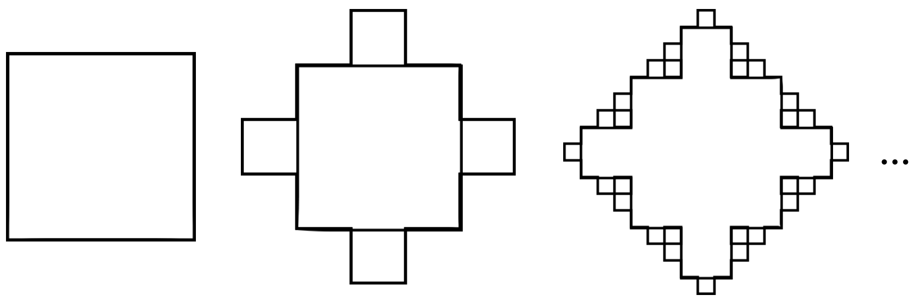
2. In the sequence of shapes shown below, how many sides does the shape in stage 4 have?



stage 0 stage 1 stage 2 …

Solution: Stage 0 has 4 sides; stage 1 has 4(5) = 20 sides; stage 2 has $4(5)^2$ = 100 sides, …
and stage 4 has $4(5)^4$ = 2500 sides.

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3. $f(x)=\dfrac{1}{1-\dfrac{1}{1-\dfrac{1}{x}}}$. Find $\left|f(2)-f(4)+f(6)\right|$.

Solution:

$$f(x)=\frac{1}{1-\dfrac{1}{1-\dfrac{1}{x}}}=\frac{1}{1-\dfrac{1}{\dfrac{x-1}{x}}}=\frac{1}{1-\dfrac{x}{x-1}}=\frac{1}{\dfrac{x-1-x}{x-1}}=\frac{1}{\dfrac{1}{1-x}}=1-x\Rightarrow\left|f(2)-f(4)+f(6)\right|=\left|-1+3-5\right|=3.$$

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4. Given the points $M(0, 220)$ and $N(n, 2020)$, where n is a positive integer. For how many values of n is the slope of MN an integer?

Solution: Slope of $MN = \dfrac{1800}{n} = \dfrac{2^3 \cdot 3^2 \cdot 5^2}{n}$, which is an integer when n is a (positive) divisor of $1800 = 2^3 \cdot 3^2 \cdot 5^2$. In all, $2^3 \cdot 3^2 \cdot 5^2$ has $4 \cdot 3 \cdot 3 = 36$ divisors $\Rightarrow$ the slope is an integer for 36 values of n.

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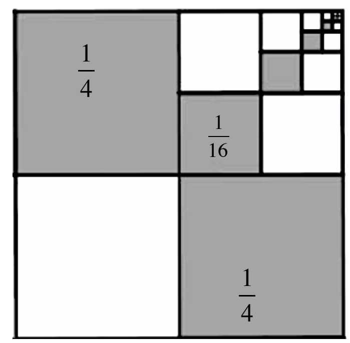
5. If the shading pattern in the top right quarter of the square continues indefinitely, what fraction of the square is shaded?

Solution: Let the area of the large square be 1.

Then, the area of the shaded region

$$= \frac{1}{2} + \frac{1}{4}\left(\frac{1}{4}\right) + \frac{1}{4}\left(\frac{1}{16}\right) + \frac{1}{4}\left(\frac{1}{64}\right) + \ldots$$

$$= \frac{1}{2} + \frac{1}{4}\left(\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \ldots\right) = \frac{1}{2} + \frac{1}{4}\left(\frac{1}{3}\right) = \frac{7}{12}.$$



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6. Let $f(x) = 2 - 3x$ and $g(x) = \dfrac{1}{x+3}$. Find the **<u>exact value</u>** of $g\big(f\big(g(23)\big)\big)$.

Solution: $g\big(f\big(g(23)\big)\big) = g\left(f\left(\dfrac{1}{26}\right)\right) = g\left(\dfrac{49}{26}\right) = \dfrac{26}{127}$.

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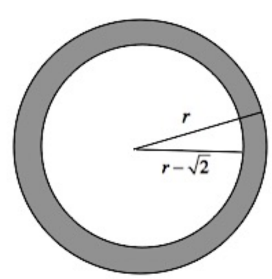
7. The shaded area between two concentric circles with radii r and $r - \sqrt{2}$ is $2\pi + 6\pi\sqrt{2}$ square units.
When the sum of the radii is written as $a + \sqrt{b}$, find $a + b$.



Solution: Area of the ring $= \pi r^2 - \pi\left(r - \sqrt{2}\right)^2 = 2\pi + 6\pi\sqrt{2} \Rightarrow 2\pi\sqrt{2} = 4 + 6\pi\sqrt{2} \Rightarrow r = \sqrt{2} + 3$.

$\therefore$ The sum of the radii $= 6 + \sqrt{2} = a + \sqrt{b} \;\Rightarrow\; a + b = 8$.

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8. The faces of two fair eight-sided dice are numbered 1 to 8. What is the probability of getting a sum of 8 when you roll the two dice once?

Solution:

Of the $8^2 = 64$ possible outcomes, only 7 show a sum of 8.

$\therefore$ The probability of getting a sum of 8 is $\frac{7}{64}$, or ≈ 0.109.

	1	2	3	4	5	6	7	8
1	2	3	4	5	6	7	8	9
2	3	4	5	6	7	8	9	10
3	4	5	6	7	8	9	10	11
4	5	6	7	8	9	10	11	12
5	6	7	8	9	10	11	12	13
6	7	8	9	10	11	12	13	14
7	8	9	10	11	12	13	14	15
8	9	10	11	12	13	14	15	16

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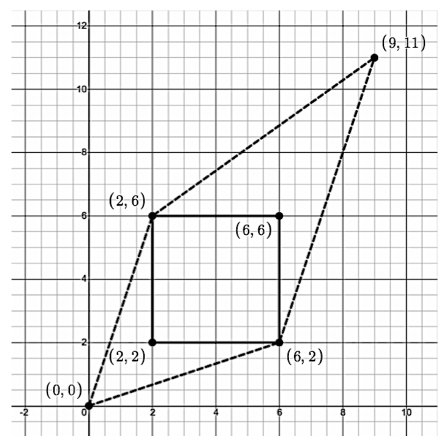
9. A robot needs to get from (0, 0) to (9, 11) on a gridded level field. The robot needs to avoid a square block that is on the field with its vertices at (2, 2), (6, 2), (2, 6) and (6, 6). If the robot can move freely in any direction on the field, find the shortest distance the robot can travel to reach its destination.

Solution: The two possible paths around the block are (0,0) to (2,6) to (9,11), which is $\sqrt{40}+\sqrt{74}$ unit, and (0,0) to (6,2) to (9,11), which is $\sqrt{40}+\sqrt{90}$ units.
$\therefore$ The shortest distance would be
$\sqrt{40}+\sqrt{74} = 2\sqrt{10}+\sqrt{74} \approx 14.927$.



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10. For what value of m do the real solutions of $x^2 - 8x + m = 0$ differ by 5?

Solution: If the solutions differ by 5, then, for the real solutions a and $a + 5$,

$$(x-a)\left(x-(a+5)\right) = x^2 - 8x + m$$

$$x^2 - x(a+5) - ax + a(a+5) = x^2 - 8x + m$$

$$x^2 - (2a+5) + a(a+5) = x^2 - 8x + m$$

$$2a + 5 = 8 \Rightarrow a = \frac{3}{2}, \text{ and } a(a+5) = \frac{39}{4} = 9.75 = m$$

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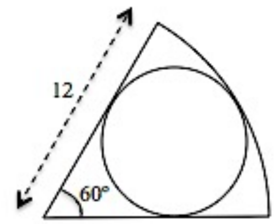
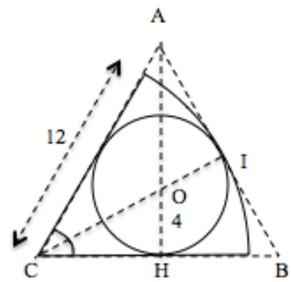
11. A circle is inscribed in a sector of another circle with a radius of 12 units and a central angle of 60º. What is the area of the inscribed circle?



Solution: Triangle ABC is equilateral, with height 12 units. The perpendicular bisectors and angle bisectors intersect at the center of the inscribed circle with center at O and radius $OH = \frac{1}{3}AH = 4$.

$\therefore$ Area of the circle = 16π.



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12. A regular hexagon and a circle have the same area. What is the ratio of the perimeter of the hexagon to the circumference of the circle?

Solution: The area of the hexagon with side a = 6(area of equilateral triangle with side a) = $\frac{3\sqrt{3}}{2}a^2$.

The area of the circle radius $r = \pi r^2$. $\frac{3\sqrt{3}}{2}a^2 = \pi r^2 \Rightarrow \frac{a^2}{r^2} = \frac{2\pi}{3\sqrt{3}} \Rightarrow \frac{a}{r} = \sqrt{\frac{2\pi}{3\sqrt{3}}}$

$$\Rightarrow \frac{\text{Perimeter of hexagon}}{\text{Circumference of circle}} = \frac{6a}{2\pi r} = \frac{3}{\pi}\sqrt{\frac{2\pi}{3\sqrt{3}}} = \sqrt{\frac{2\sqrt{3}}{\pi}} \approx 1.050$$

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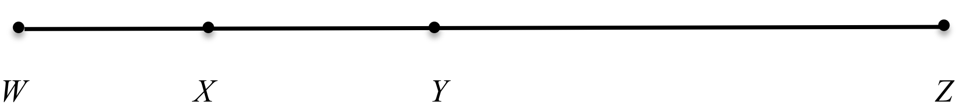
13. Points X and Y are on a line segment $\overline{WZ}$, with Y between X and Z as shown below.



If $\frac{WX}{XZ} = \frac{1}{5}$ and $\frac{XY}{YZ} = \frac{2}{3}$, what is the value of $\frac{XY}{WZ}$?

Solution: Let $WZ = 1$. Then, $WX + XZ = 1$ and $5WX = XZ \Rightarrow WX = \frac{1}{6}$ and $XZ = \frac{5}{6}$.

$XZ = \frac{5}{6} \Rightarrow XY + YZ = \frac{5}{6}$. Also, $3XY = 2YZ \Rightarrow YZ = \frac{1}{2}$ and $XY = \frac{1}{3}$.

This means $\frac{XY}{WZ} = \frac{1}{3}$ for any length WZ.

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14. What is the 2020^{th} digit after the decimal in the expansion of $\frac{1}{41}$?

Solution: $\frac{1}{41} = 0.\overline{02439}$ and $2020 = 5 \times 404$. Therefore, the 2020^{th} digit after the decimal is 9.

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15. In a certain type of 3×3 array, the entries can be 0 or 1 in such a way that there is no more than one 0 in each row and each column.

For example, the array $\begin{matrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{matrix}$ is acceptable, whereas $\begin{matrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{matrix}$ is not acceptable.

How many acceptable 3×3 arrays are possible?

Solution: The acceptable arrays must have no, one, two or three 0's. There is only 1 array with no zeros; 9 with one 0; 18* with two 0's ; and 6** with three 0's for a total of 34.

* $\frac{9 \times 4}{2!} = 18$ (9 possiblities for the first 0, and 4 for the second 0, and divide by 2! to account for repeated patterns.)

** $\frac{9 \times 4}{3!} = 6$ (9 possiblities for the first 0, 4 for the second 0, which leaves only one possible option for the third, and divide by 3! to account for repeated patterns.)

The 18 unique arrays with 2 zeros:

$$\begin{matrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{matrix}, \begin{matrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{matrix}, \begin{matrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{matrix}, \begin{matrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{matrix}, \begin{matrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{matrix}, \begin{matrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{matrix}, \begin{matrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{matrix}, \begin{matrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{matrix}, \begin{matrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{matrix}$$

$$\begin{matrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{matrix}, \begin{matrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{matrix}, \begin{matrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{matrix}, \begin{matrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{matrix}, \begin{matrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{matrix}, \begin{matrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{matrix}, \begin{matrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{matrix}, \begin{matrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{matrix}, \begin{matrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{matrix}$$

The 6 unique arrays with 3 zeros:

$$\begin{matrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{matrix}, \begin{matrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{matrix}, \begin{matrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{matrix}, \begin{matrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{matrix}, \begin{matrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{matrix}, \begin{matrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{matrix}$$