39th Annual AMTNJ High School Mathematics Contest

Answer Key

1. 16

2. 2,520

3. $\frac{3}{14}$

4. (2, 7)

5. 1.2%

6. 57.834

7. $3 + \sqrt{13}$

8. $27 - \frac{9\pi}{2}$
or 12.862
or 12.863

9. 765,756

10. $-\sqrt[3]{9}$ or $-9^{\frac{1}{3}}$ or $-3^{\frac{2}{3}}$
or $\sqrt[3]{-9}$

11. 17

12. $24 + 4\pi$

13. 10

14. 1,015,560

15. $\frac{1140}{13}$ or 87.692

1. How many ordered pairs (a, b) of non-negative integers are there that satisfy $5a + 4b \leq 20$?

Solution: There are 16 ordered pairs that can be counted by graphing *b* vs. *a.*

$(0,0), (1,0), (2,0), (3,0), (4,0)$

$(0,1), (0,2), (0,3), (0,4), (0,5)$

$(1,1), (1,2), (1,3), (2,1), (2,2), (3,1)$

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2. Find the smallest positive integer that is divisible by every positive integer less than 10.

Solution: The smallest such integer is $5 \times 7 \times 2^3 \times 3^2 = 2{,}520$

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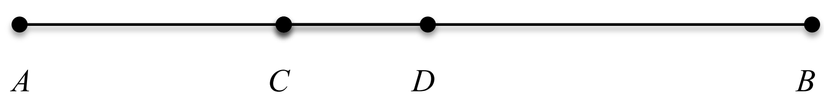
3. Points *C* and *D* are on the line segment $\overline{AB}$ of fixed length, with *D* between *C* and *B* as shown. Note: The diagram is not drawn to scale.



If $\dfrac{AC}{CB} = \dfrac{1}{3}$ and $\dfrac{CD}{DB} = \dfrac{2}{5}$, what is the value of the ratio $\dfrac{CD}{AB}$?

Solution 1: Let the length of $\overline{AB}$ be 1. $AC + CB = 1 \; and \; 3AC = CB \Rightarrow AC = \frac{1}{4} \; and \; CB = \frac{3}{4}$.

$5CD = 2DB \; and \; CB = \frac{3}{4} \Rightarrow CD = \frac{3}{14}$. Therefore, $\frac{CD}{AB} = \frac{3}{14}$. This result is generalized to any length of the segment $\overline{AB}$.

Solution 2: If $AC = x, CB = 3x, CD = 2y$, and $DB = 5y$, then $7y = 3x$, and $\frac{CD}{AB} = \frac{2y}{4x} = \frac{y}{2x} = \frac{\frac{3}{7}x}{2x} = \frac{3}{14}$

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4. Find the coordinates of the point on the line $y = 2x + 3$ that is equidistant from, and closest to the two points $(14, 1)$ and $(-10, 13)$.

Solution: If the two points are equidistant from and closest to the line, then their midpoint must be on the line. The midpoint is (2, 7)

5. The amount in grams of radioactive isotope bromine-77 remaining after t days is modeled by the exponential function $A(t) = A_0(0.5)^{\frac{t}{57}}$, where A_0 is the amount in grams of the isotope present at $t = 0$. What is the daily decay rate of bromine-77? Write your answer as a percent rounded to the nearest tenth.

Solution 1 : Rewrite the exponential function in the form $A_0(1-r)^t$ to reveal r, the daily decay rate.

$$A(t) = A_0\left(0.5^{\frac{1}{57}}\right)^t = A_0(0.9879)^t = A_0(1-.0121)^t \Rightarrow r = 1.2\%$$

Solution 2: Using $A(t) = A_0e^{-rt}$ as the exponential decay model, the decay rate is

$$r = -\ln(0.5)^{\frac{1}{57}} = 0.012 = 1.2\%$$

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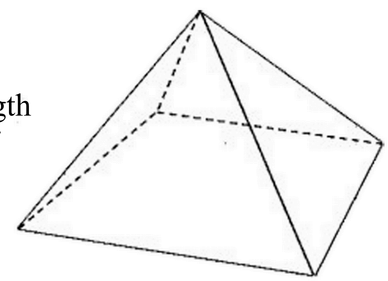
6. The base of a rectangular pyramid is 26 square units, and its volume is 54 cubic units. If the width of the base is increased by 20%, the length decreased by 15% and the height increased by 5%, find the volume of the new pyramid.



Solution: $V = \frac{1}{3}(Area\ of\ base)(height) \Rightarrow height = \frac{81}{13}\ units.$

$Area\ of\ base = 26 \Rightarrow length = \frac{26}{width}$

$New\ Volume = \frac{1}{3}(1.2w)\left(.85\times\frac{26}{w}\right)\left(1.05\times\frac{81}{13}\right) = 57.834\ cubic\ units.$

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7. Let x and y be positive real numbers. If $x = 3+\cfrac{1}{3+\cfrac{1}{x}}$ and $y = 3+\cfrac{1}{3+\cfrac{1}{3+\cfrac{1}{y}}}$,

find the exact value of $x+y$.

Solution: $x = 3+\cfrac{1}{3+\cfrac{1}{x}} \Rightarrow x^2-3x-1=0 \Rightarrow x=\frac{3+\sqrt{13}}{2}$ (Note that x is positive and cannot equal $\frac{3-\sqrt{13}}{2}$). Similarly, $y=\frac{3+\sqrt{13}}{2}$. Therefore, $x+y=3+\sqrt{13}$.

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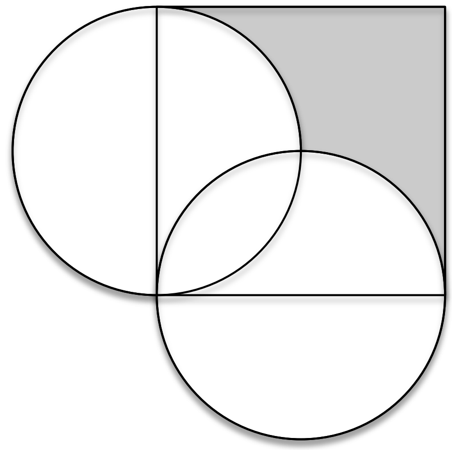
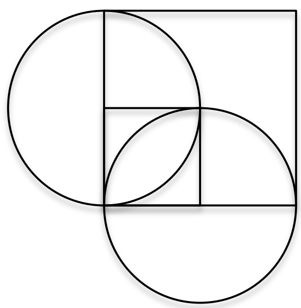
8. The side of the square shown on the right has length 6. The two circles are tangent to the two sides of the squares, and their diameters coincide with the other two sides. Find the area of the shaded region that is inside the square but lies outside the two circles.



Solution:



Area of shaded region = Area of large square – Area of small square – 2(Area of 1/4th of circle)

$$= 6^2 - 3^2 - 2\left(\frac{\pi(3^2)}{4}\right) = 27 - \frac{9\pi}{2} \approx 12.8628$$

9. Fifteen students from the environmental club at a high school volunteered to clean a local park. The club advisor would like to have three teams of five students each. If each team is to be assigned to a different section of the park, in how many different ways can the club advisor assign the 15 students to the three groups?

Solution: There are $\binom{15}{5} = 3003$ ways of choosing the first team of 5, which leaves 10 students.

There are $\binom{10}{5} = 252$ ways of choosing the second team, and the remaining students would make up the third. Therefore, there are (3003)(252) = 765,756 different ways of assigning 15 students in three groups of five.

10. What is the exact value of k for which $(x+2)$ is a factor of $4x^4-(kx)^3-8x-8$?

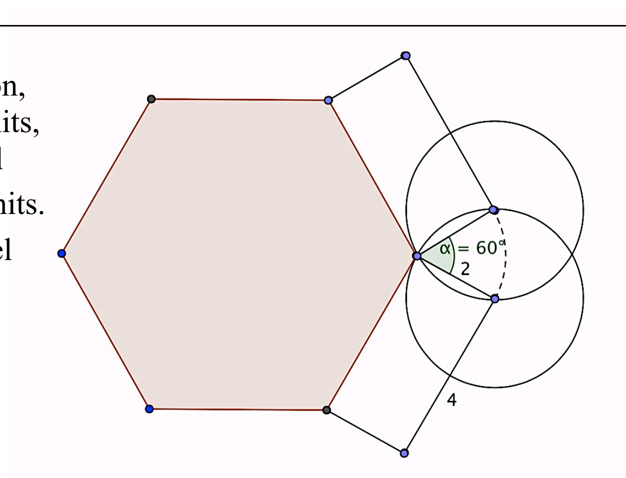
Solution: If $x+2$ is a factor, then $4(-2)^4-k^3(-2)^3-8(-2)-8=0$. Therefore, $k=-\sqrt[3]{9}$.

11. A *palindrome* is a number that reads the same from left to right and from right to left. 2, 22 and 232 are examples of palindromes. How many palindromes between 1 and 1000 are divisible by 11?

Solution: There are $\boxed{17}$ such palindromes: 11, 22, 33, 44, 55, 66, 77, 88, 99, 121, 242, 363, 484, 616 (= 65x11), 737 (= 76x11), 858 (= 87x11), and 979 (= 98x11) .

12. A circle with radius 2 units rolls along the outside of a regular hexagon of side 4 units. Find the exact value of the total distance traveled by the center of this circle after it completes its trip along the outside of the hexagon exactly once, and returns to its starting position.

Solution: Along each of the edges of the hexagon, the center of the circle will travel 4 units, and around each of the vertices, it will traverse an arc equal in length to $\frac{2\pi}{3}$ units. The total distance the center will travel is $\boxed{24+4\pi}$ units.



13. Let v and w be non-negative integers less than or equal to 10. For how many ordered pairs (v,w) is the expression $(v+w)^2+(vw-1)^2$ a prime number?

Solution: $(v+w)^2+(vw-1)^2=v^2+w^2+v^2w^2+1=(v^2+1)(w^2+1)$ which is prime when $v^2+1=1$ *and* w^2+1 is prime, or when $w^2+1=1$ *and* v^2+1 is prime. When $v=0$, $w=1,2,4,6,$ *and* 10 will make the expression prime. Similarly, when $w=0, v=1,2,4,6$ *and* 10 will make the expression prime. This brings the total of ordered pairs to $\boxed{10}$.

14. Let the sequence a_n have the property that, for any nonnegative integer *k,* $\sum_{i=4k+1}^{4(k+1)} a_i = 2015$.

Evaluate $\sum_{i=1}^{2016} a_i$.

Solution: The sum consists of 504 groups of $\sum_{i=4k+1}^{4(k+1)} a_i$, therefore its value is 504 • 2015 = 1,015,560.

15. Given the rectangle *ABCD* in which *AB* = 10 and *BC* = 12. If *M* is the midpoint of side *CD* and *K* is a point on *AM* such that *AK=AD,* what is the area of quadrilateral *ABCK* ?

Solution 1: Area of *ABCK* = Area of *ABCD* – Area of $\triangle ADM$ – Area of $\triangle CKM$

= 120 – 30 – A of $\triangle CKM$.

To find the area of $\triangle CKM$, note that *AM* = 13, *KM* = 1, and area of $\triangle AMC$ = ½(60)=30.
Moreover, using sides *AM* and *KM* for base, triangles *AMC* and *KMC* share a height.
This means that the area of $\triangle CKM = \frac{1}{13}$ (area of $\triangle AMC$) ⇒ Area of $\triangle CKM = \frac{30}{13}$.
Therefore, area of $ABCK = \frac{1140}{13}$ or 87.692.

Solution 2: $\sin(\angle AMD) = \sin(\angle AMC) = \frac{12}{13}$ (the two angles are supplementary).

Area of $\triangle\ KMC = \frac{1}{2} \bullet 5 \bullet 1 \bullet \sin(\angle KMC) = \frac{30}{13}$.

∴ Area of *ABCK* = Area of *ABCM* – Area of $\triangle\ KMC = \frac{1}{2} \bullet 12 \bullet 15 - \frac{30}{13} = \frac{1140}{13} = 87.692.$