40th Annual AMTNJ High School Mathematics Contest

Answer Key

1. 6	**6.** 5	**11.** 15
2. 8	**7.** $-1+\sqrt{3}$	**12.** 24
3. 111	**8.** $\frac{1}{4}$	**13.** 3
4. $\frac{13}{14}$	**9.** $\frac{12}{11}$ *or* $1\frac{1}{11}$	**14.** $36\sqrt{2}$
5. $\sqrt[3]{-6}$ *or* $-\sqrt[3]{6}$	**10.** 1032	**15.** 8008

1. Find the area of the triangle that has vertices (–1, 3), (1, 1) and (3, 5).

Solution 1: This is an isosceles triangle with height $\sqrt{3^2+3^2}=3\sqrt{2}$ and base length $\sqrt{2^2+2^2}=2\sqrt{2}$.

The area = $\frac{1}{2}\cdot 3\sqrt{2}\cdot 2\sqrt{2}=6$.

Solution 2: Using determinants, the area is $\frac{1}{2}|\det A|$ = 6, where $A=\begin{bmatrix} -1 & 3 & 1 \\ 1 & 1 & 1 \\ 3 & 5 & 1 \end{bmatrix}$.

---

2. Given three consecutive odd integers *a, b* and *c* such that $0<a<b<c$. What is the value of $a^2-2b^2+c^2$?

Solution: The integers are consecutive and odd, and $0<a<b<c \Rightarrow b=a+2, c=a+4$.

$$\therefore\ a^2-2b^2+c^2=a^2-2(a+2)^2+(a+4)^2$$
$$=a^2-2a^2-8a-8+a^2+8a+16$$
$$=8$$

---

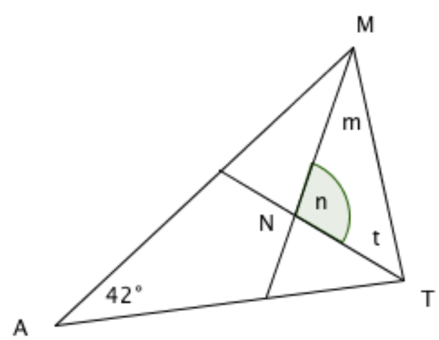
3. The measure of angle *A* in a scalene triangle *AMT* is 42°. The bisectors of angles *M* and *T* intersect in point *N* inside the triangle. Find the measure of angle *MNT.*

Solution: In $\triangle AMT$, $42^\circ+2m+2t=180^\circ$

$\Rightarrow m+t=69^\circ$

$\Rightarrow n=180^\circ-(m+t)$

$\Rightarrow n=111^\circ$



---

4. What is the exact value of $\frac{3^{2017}-3^{2014}}{3^{2017}+3^{2014}}$ in its simplest form?

Solution: $\frac{3^{2017}-3^{2014}}{3^{2017}+3^{2014}}=\frac{3^{2014}\left(3^3-1\right)}{3^{2014}\left(3^3+1\right)}=\frac{26}{28}=\frac{13}{14}$.

5. Find the <u>exact value</u> of k if $\dfrac{2x^4-(kx)^3+x^2-4}{x+1} = p(x) - \dfrac{7}{x+1}$ for some polynomial $p(x)$.

Solution 1: When $2x^4-(kx)^3+x^2-4$ is divided by $x+1$, the remainder is -7.
By the Remainder Theorem,

$$2(-1)^4-(-k)^3+1-4=-7$$
$$\Rightarrow k^3=-6$$
$$\Rightarrow k=\sqrt[3]{-6} \; or \; -\sqrt[3]{6}$$

Solution 2: Using long or synthetic division,

$$\begin{array}{r|rrrrr} -1 & 2 & -k^3 & 1 & 0 & -4 \\ & & -2 & k^3+2 & -k^3-3 & k^3+3 \\ \hline & 2 & -k^3-2 & k^3+3 & -k^3-3 & k^3-1=-7 \end{array} \Rightarrow k=\sqrt[3]{-6}$$

---

6. Given a right triangle with legs that have lengths *m* and *n,* and a hypotenuse of length *n* + 1.
If *m* and *n* are integers and $n \le 60$, how many such triangles exist?

Solution: $n^2+m^2=(n+1)^2$ which means $m^2=2n+1$, where $n\le 60$. This means m^2 is odd and can be $9(n=4)$, $25(n=12)$, $49(n=14)$, $81(n=40)$ or $121(n=60)$. Therefore, there are only 5 such triangles.

---

7. Find the exact value of the infinite continued fraction $\dfrac{2}{2+\dfrac{2}{2+\dfrac{2}{2+\dfrac{2}{2+\dots}}}}$.

Solution: Let $x=\dfrac{2}{2+\dfrac{2}{2+\dfrac{2}{2+\dfrac{2}{2+\dots}}}}$. Then, $x=\dfrac{2}{2+x} \Rightarrow x^2+2x-2=0 \Rightarrow x=-1+\sqrt{3}$.

Note that $x=-1-\sqrt{3}<0$ and cannot be the value of this positive expression.

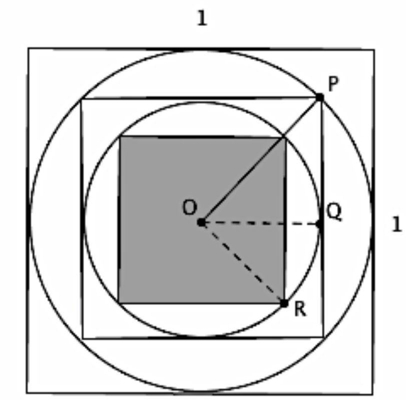
8. Find the area of the shaded inner square in the diagram shown, where the outer square has side 1, and the two circles are tangent to the sides of the squares.

Solution: $OP = \frac{1}{2} \Rightarrow$ in right $\triangle OPQ$, $2(OQ)^2 = \left(\frac{1}{2}\right)^2 \Rightarrow OQ = \frac{1}{2\sqrt{2}}$.

$OQ = OR \Rightarrow$ diagonal of the shaded square = $\frac{1}{\sqrt{2}}$

$\Rightarrow$ side of the shaded square = $\frac{1}{2}$

$\Rightarrow$ Area of the shaded square = $\frac{1}{4}$



---

9. An *arithmetic sequence* is a sequence of numbers that differ by a constant, called the *common difference*. Given an arithmetic sequence in which the sum of the first *n* terms is 216. If the first term in this sequence is *n* and the nth term is 2*n*, find the common difference of the sequence.

Solution: Let a_n represent the n$^{\text{th}}$ term of the sequence. Then, $a_1 = n$ and $a_n = 2n$.

The sum of the first n terms in the sequence = $\frac{\textit{number of terms}}{2}(\textit{first term} + \textit{last term})$

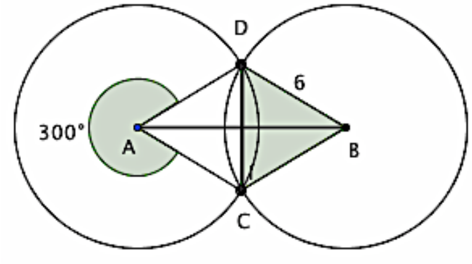
Therefore, $\frac{n}{2}(n+2n) = \frac{3n^2}{2} = 216 \Rightarrow n = 12$. Also, $a_n = a_1 + (n-1)d$, where d is the common difference. This means that $2n = n + (n-1)d$, thus $d = \frac{12}{11}$ *or* $1\frac{1}{11}$.

---

10. Two congruent circles with centers *A* and *B* have a radius of length 6 units. The circles intersect at points *C* and *D*, and chord $\overline{CD}$ also has a length of 6 units. If the area of the region enclosed by the two circles is written as $a\pi + \sqrt{b}$, what is the value of $a+b$?

Solution: $\triangle ACD$ and $\triangle BDC$ are equilateral.

The area of each sector with central angle 300° and radius 6 units is $\frac{5}{6}$th of the area of each circle, or $\frac{5}{6}(36\pi) = 30\pi$. The area of each equilateral triangle with side 6 is $\frac{1}{2}\left(\frac{6\sqrt{3}}{2}\right)(6) = 9\sqrt{3}$.



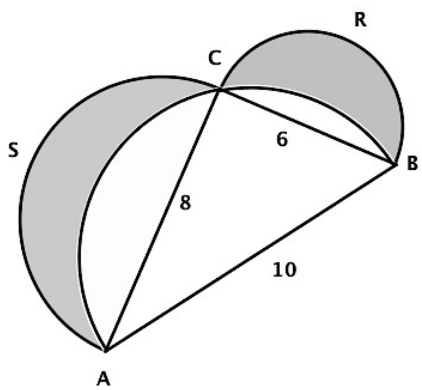
Total Area = $60\pi + 18\sqrt{3} = 60\pi + \sqrt{972} = a\pi + \sqrt{b}$. $\therefore$ $a + b = 1032$.

11. A *partition* of a positive integer n is a way of writing n as the sum of positive integers. When counting partitions, rearrangements of integers in a sum are counted once. For example, 4 has 5 partitions: 4, 3+1, 2+2, 2+1+1, and 1+1+1+1. Note that 2+1+1, 1+2+1 and 1+1+2 are counted as one partition. How many partitions does 7 have?

Solution: There are 15 partitions of 7. They are 7, 6+1, 5+1+1, 5+2, 4+1+1+1, 4+2+1, 4+3, 3+1+1+1+1, 3+2+1+1, 3+2+2, 3+3+1, 2+1+1+1+1+1, 2+2+1+1+1, 2+2+2+1, and 1+1+1+1+1+1+1.

---

12. Triangle ABC is inscribed in a semicircle with diameter $\overline{AB}$, with $AB = 10$, $AC = 8$ and $CB = 6$. If semicircles ASC and CRB have diameters $\overline{AC}$ and $\overline{CB}$ respectively, what is the area of the shaded region in the figure on the right?



Solution: The area inside semicircle ACB and outside triangle ABC = $\frac{25\pi}{2} - 24$, and the areas of the semicircles ASC and CRB are 8π and $\frac{9\pi}{2}$ respectively.

Therefore, the area of the shaded region is $8\pi + \frac{9\pi}{2} - \left(\frac{25\pi}{2} - 24\right) = 24$.

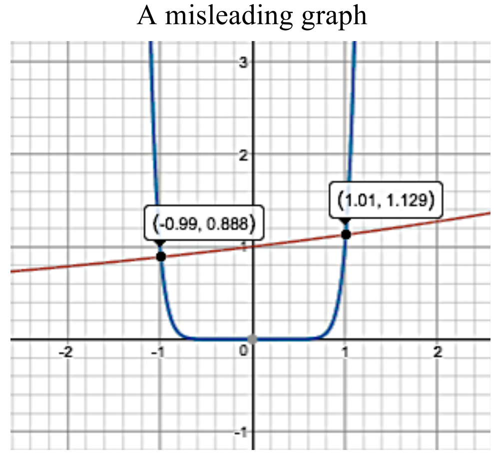
---

13. How many real solutions does the equation $e^{0.12x} - x^{12} = 0$ have?

Solution: This equation has as many real solutions as there are points of intersection between the graphs of the exponential function $f(x) = e^{0.12x}$ and the power function $g(x) = x^{12}$. These curves intersect exactly 3 times. The sketch on the right shows two points of intersection; however, this is misleading. For some large real number x, the value of the exponential function will exceed the value of the power function, and this will only be possible if the two curves intersect a third time.

(The three solutions are: –0.99, 1.01 and 647.28)

A misleading graph

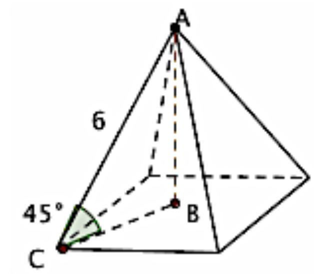


14. In a regular square pyramid, a lateral edge is 6 cm long. Find the exact volume of the pyramid if each lateral edge makes a 45° angle with its projection in the base.

Solution: $AB = CB = 3\sqrt{2}$, the diagonal of the square base has length $6\sqrt{2}$, and the side of the base is 6 units.

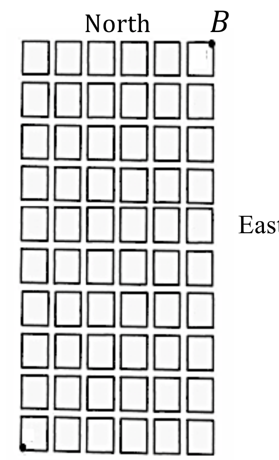
The volume of the pyramid = $\frac{1}{3}(\textit{area of the base})(\textit{height})$

$$= \frac{1}{3} \cdot 36 \cdot 3\sqrt{2} = 36\sqrt{2}.$$



---

15. Sections of Portland, Oregon are laid out in rectangular blocks. One such section consists of eleven east-west streets intersecting with seven north-south streets as shown. Find the number of different paths, each 16 blocks long, in going from the south-west corner *A,* to the north-east corner *B.*



Solution: Since each path is exactly 16 blocks long, the paths will be a combination of 10 northbound street segments (N_i) and 6 eastbound street segments (E_j). The total number of paths is the number of different arrangements of N_i's and E_j's in 16 slots.

In all, there are ${}_{16}C_6 = {}_{16}C_{10} = \frac{16!}{10!6!} = 8008$ different paths to get from point *A* to point *B*.
