# 41st Annual AMTNJ High School Mathematics Contest

## Answer Key

**1.** 45

**2.** 53

**3.** $\frac{31}{5}$ or $6\frac{1}{5}$ or 6.2

**4.** 46

**5.** 28

**6.** $\frac{2}{9}$ or $0.\overline{2}$ or 0.222

**7.** $1+\sqrt{3}$

**8.** 5,636

**9.** $\sqrt{30}$

**10.** 11

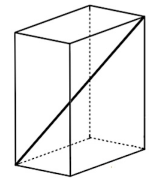
**11.** 360

**12.** $\frac{2}{9}$ *or* $0.\overline{2}$ *or* 0.222

**13.** 4

**14.** 12

**15.** $\frac{50\sqrt{3}-25\pi}{2}$ *or* $25\sqrt{3}-12.5\pi$



1. The dimensions *a, b* and *c* of a rectangular prism are in the ratio 3:5:7. If the length of the diagonal is $3\sqrt{83}$ units, what is the value of $a+b+c$?

Solution: The length of the diagonal = $\sqrt{a^2+b^2+c^2} = \sqrt{(3x)^2+(5x)^2+(7x)^2} = \sqrt{83x^2} = 3\sqrt{83}$.
Therefore $x = 3$, and the sum = 9 + 15 + 21 = 45.

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2. Given positive integers *m* and *n* that differ by 35, such that $m^2+n^2=2017$.
Find the value of $m + n$.

Solution 1: $(m-n)^2 = 1225 = 2017-2mn \Rightarrow mn = 396 \Rightarrow (m+n)^2 = 2017+2mn = 2809 \Rightarrow m+n = 53$.

Solution 2: $m = n+35 \text{ and } m^2+n^2 = 2017 \Rightarrow n^2+35n-396=0 \Rightarrow n=9 \Rightarrow m+n=53$.

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3. Find the *y*-coordinate of the point on the line $y=2x+3$ that is closest to the point $(0,7)$.

Solution 1: Let $(a, 2a+3)$ be that closest point. Then, the distance $\sqrt{(2a+3-7)^2+(a-0)^2}$ is minimum when $(2a-4)^2+a^2 = 5a^2-16a+16$ is minimum $\Rightarrow a=\frac{8}{5}$, and the *y*-coordinate $=2\left(\frac{8}{5}\right)+3=\frac{31}{5}$.

Solution 2: The closest point on the line is the intersection between the line and the perpendicular to it that passes through (0,7). The perpendicular has equation $y=-\frac{1}{2}x+7$, and the intersection is $\left(\frac{8}{5},\frac{31}{5}\right)$.

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4. Some values of a function *f* are shown in the following table:

| $x$ | –5 | –3 | –1 | 1 | 3 | 5 | 7 | 9 |
|---|---|---|---|---|---|---|---|---|
| $f(x)$ | –54 | –20 | –2 | 0 | –14 | –44 | –90 | –152 |

If $g(x) = -\frac{2}{3}f(1-2x)+f(|x|)$, what is the value of $g(-3)$?

Solution: $g(-3) = -\frac{2}{3}f(7)+f(3) = -\frac{2}{3}(-90)-14 = 46$.

5. What is the largest possible number of intersections between eight distinct lines in a plane?

Solution: The number of points will be maximum when the lines are not parallel, and no more than two lines intersect in the same point. $n$ lines intersect in a most $\frac{n(n-1)}{2}$ points.

The following table shows the pattern for these triangular numbers:

| Number of lines | Maximum points of intersection |
|---|---|
| 2 | 1 |
| 3 | 1 + 2 = 3 the third line intersects the two others |
| 4 | 1 + 2 + 3 = 6 the 4th line intersects the three others |
| 5 | 1 + 2 + 3 + 4 = 10 |
| …. | …. |
| 8 | 1 + 2 + 3 + 4 + 5 + 6 + 7 = 28 |

6. If you randomly arrange 5 different math books and 5 different art books on a shelf, what is the probability that you will have two math books at the two ends?

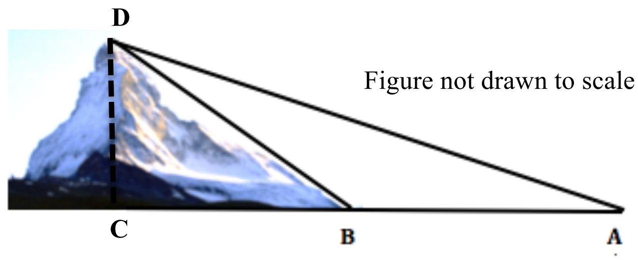
Solution: Regardless of the arrangement of the eight books in the middle, we need to consider the two ends. There are ${}_{10}C_2 = 45$ different ways of selecting any two books, and ${}_{5}C_2 = 10$ different ways of choosing two math books.

Therefore, the probability of having two math books at the ends is $\frac{10}{45} = \frac{2}{9}$.

7. Find the <u>exact value</u> of the infinite continued fraction $2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+\ldots}}}}$.

Solution: Let $x = 2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+\ldots}}}}$. Then, $x = 2+\cfrac{1}{1+\cfrac{1}{x}} \Rightarrow x-2 = \frac{x}{x+1} \Rightarrow x^2-2x-2=0 \Rightarrow x = 1+\sqrt{3}$.

Note that $x = 1-\sqrt{3} < 0$ and cannot be the value of this positive expression.




8. The peak of a mountain can be seen from points A and B that are both 5,200 feet above sea level, and are 500 feet apart. The angle of elevation at A and B are 20° and 32° respectively. Find the elevation from sea level at the mountain peak. Round your answer to the nearest foot.

Solution: $CD = BC\tan 32 = (500 + BC)\tan 20 \Rightarrow BC = \dfrac{500\tan 20}{\tan 32 - \tan 20} \approx 697.53$

$CD = 697.53\tan 32 \approx 435.865 \Rightarrow$ peak elevation = 5,635.865 ≈ 5,636 feet above sea level.

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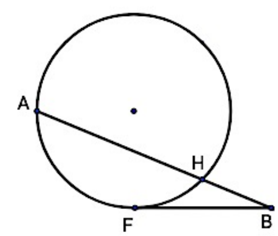
9. In the figure shown on the right, $\overline{FB}$ is tangent to the circle with center $C$, and $\overline{AB}$ intersects the circle in point $H$. If $AH$ = 7cm and $HB$ = 3cm, find the exact length of $\overline{FB}$.




Solution: $BA \cdot BH = BF^2 \Rightarrow BF = \sqrt{30}$ .

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10. How many integer solutions does the inequality $|n-9| < 6$ have?

Solution 1: $n - 9$ could be $0, \pm1, \pm2, \pm3, \pm4, \pm5$, and these lead to 11 integer solutions to $|n-9| < 6$.
Solution 2: There are 11 integers for which $3 < n < 15$.

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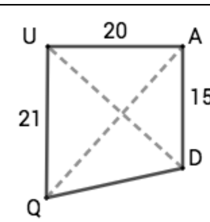
11. In quadrilateral $QUAD$, $QU$=21, $UA$=20, $AD$=15, $UD$=25 and $AQ$=29. Find the area of $QUAD$.

Solution: $QU^2+UA^2=AQ^2$ and $UA^2+AD^2=UD^2 \Rightarrow QUA$ and $UAD$ are right triangles.
$\therefore$ $QUAD$ is a right trapezoid, and its area is $\frac{1}{2}(15 + 21) \cdot 20 = 360$.

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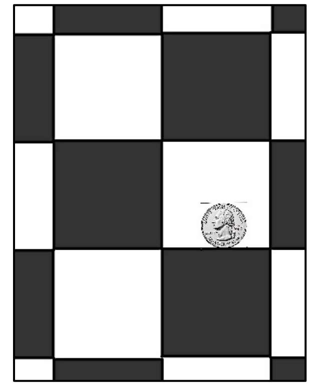
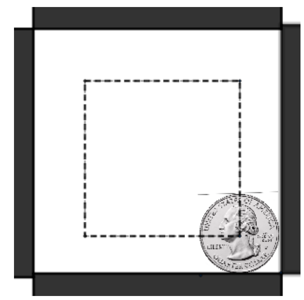
12. A coin with diameter 1 inch is tossed onto a $24 \times 18$ in$^2$ flat surface covered with black and white square tiles as shown. The side of each tile is 3 inches.

What is the probability that the coin lands entirely inside a white tile?

Here's a partial view of the surface:



Solution: The coin will be entirely inside a 3x3 square when the center of the coin is inside a 2x2 square as shown. Therefore, the probability of the coin landing inside a square of side 3 is $\frac{Area\ of\ small\ square}{Area\ of\ large\ square} = \frac{4}{9}$ .
There are half as many white squares as black squares on the surface,
Therefore, the probability of landing on a white square is $\frac{1}{2} \cdot \frac{4}{9} = \frac{2}{9}$ .



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13. How many points do the graphs of $f(x) = x^3 + 1$ and $g(x) = e^{0.3x}$ have in common?

Solution: The curves intersect exactly 4 times. The three intersection points are visible when the functions are graphed in a small window such as [–2,2] x [0,2]; however, the value of the exponential function will exceed the value of the power function, and this will only be possible if the two curves intersect a fourth time.
The curves intersect when $x = 0$ and $x \approx -0.527,\ 0.572$, and 35.772.

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14. A *look and say sequence* is a sequence of integers that begins with a single digit, in which the next term is obtained by describing the previous term.
For example, the sequence starting with the single digit 1 is $1, 11^{*}, 21^{**}, 1211^{***}, ...$

$^{*}$there is one 1 in the previous term,
$^{**}$there are two 1's in the previous term,
$^{***}$there is one 2 and one 1 in the previous term.
The fourth term of this sequence has 4 digits.

How many digits are in the seventh term of the *look and say sequence* that begins with 2?

Solution: The sequence is 2, 12, 1112, 3112, 132112, 1113122112, 311311222112, …
in which the seventh term has 12 digits.

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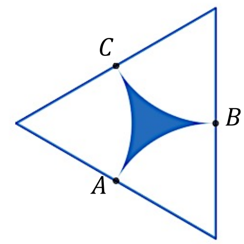
15. The image shows an equilateral triangle with perimeter 30 units. Each circular arc is centered at a vertex, and has radius that is half the length of each side. Find the exact value of the area of the shaded region with vertices *A, B,* and *C.*




Solution: Area of the equilateral triangle = $\frac{1}{2}\left(5\sqrt{3}\right)(10) = 25\sqrt{3}\,u^2$

Area of the three circular sectors = $3\left(\frac{1}{6}\pi(5)^2\right) = \frac{25\pi}{2}u^2$

$\therefore$ Area of deltoid $ABC = \dfrac{50\sqrt{3} - 25\pi}{2}\,u^2$