42nd Annual AMTNJ High School Mathematics Contest

Answer Key

1. 16

2. 4

3. $\sqrt{69}$

4. 10

5. 341

6. 320

7. $\dfrac{37}{64}$

8. 33.69º

9. $\dfrac{3}{2}$ or 1.5

10. $-\dfrac{55}{4}$ or $-13\dfrac{3}{4}$ or -13.75

11. $\dfrac{4}{7}$ or 0.571

12. 1,440

**13.** 32

14. 22

15. 1.7

1. Find the area of the region in the xy-plane bounded by the three lines $y = -\frac{5}{4}x$, $y = \frac{3}{4}(x-4)-5$ and $x = 0$.

Solution: The region is the triangle with vertices (0, 0), (4, –5) and (0, –8) and its area $= \frac{1}{2} \cdot 4 \cdot 8 = 16$.

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2. If $\dfrac{2}{x^2 - 3x + 2} = \dfrac{n}{x-2} - \dfrac{m}{x-1}$ for integers n and $m,$ and real numbers $x,$ $(x \neq 1, 2),$ find the value of $m + n.$

Solution: $\dfrac{2}{x^2 - 3x + 2} = \dfrac{n}{x-2} - \dfrac{m}{x-1} \Rightarrow 2 = n(x-1) - m(x-2) \Rightarrow 0x + 2 = x(n-m) - n + 2m$

$\Rightarrow n - m = 0$ and $-n + 2m = 2 \Rightarrow n = m = 2$, and $m + n = 4$.

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3. The first seven triangles in a sequence of right triangles are shown.
    The first triangle has legs that are 1 and 2 units long. All other triangles
    have one leg that is 2 units long, and another leg that is the hypotenuse
    of the previous triangle in the sequence. What is the <u>exact length</u> of
    the hypotenuse of the seventeenth triangle in this sequence?
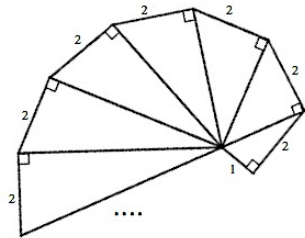
Solution: The lengths of the hypotenuse form the sequence $\sqrt{5}, \sqrt{9}, \sqrt{13}, \sqrt{17}, \ldots, \sqrt{4n+1}, \ldots$

For the seventeenth triangle, $n = 17$ and the hypotenuse $= \sqrt{4(17)+1} = \sqrt{69}$.

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4. If f is defined such that $f(2) = 1$ and $f(2n) = n + f(2n-2)$ for positive integers n greater than 1, find $f(8)$.

Solution: $f(8) = f(2 \cdot 4) = 4 + f(2(4) - 2)$

$= 4 + f(6) = 4 + f(2 \cdot 3) = 4 + 3 + f(2(3) - 2)$

$= 4 + 3 + f(4) = 4 + 3 + f(2 \cdot 2) = 4 + 3 + 2 + f(2(2) - 2)$

$= 4 + 3 + 2 + f(2) = 4 + 3 + 2 + 1 = 10$

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5. The curve defined by $x + \dfrac{3}{y-4} = 7$ intersects the coordinate axes at $(a, 0)$ and $(0, b)$.

    If $a + b = \dfrac{B}{28}$, find the value of B.

Solution: $x = 0 \Rightarrow y = \dfrac{31}{7}$, $y = 0 \Rightarrow x = \dfrac{31}{4}$.

$\dfrac{31}{7} + \dfrac{31}{4} = \dfrac{31(4+7)}{28} = \dfrac{341}{28} \Rightarrow B = 341$.

6. Your friend starts tapping his desk to a tune with four beats using five fingers. How many different fingering sequences with four beats are possible if he avoids double tapping?
For example, if you label the pinky P, ring finger R, middle M, index I and thumb T, the sequence PMPI counts, but PPMI doesn't count because P is a double.

P R M I   T

Solution: There are 5 finger choices for the first note, and 4 choices each for the other three notes, resulting in $5 \times 4^3 = 320$ fingering possibilities.

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7. If the <u>unit square</u> is divided into smaller squares as shown on the right, what is the area of the shaded region?
Write your answer as a <u>fraction in reduced form</u>.
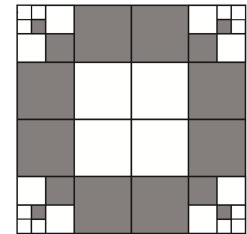
Solution: Total shaded area $= 8\left(\dfrac{1}{4^2}\right) + 4\left(\dfrac{1}{8^2}\right) + 4\left(\dfrac{1}{16^2}\right) = \dfrac{1}{2} + \dfrac{1}{16} + \dfrac{1}{64} = \dfrac{37}{64}$.

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8. What is the degree measure of the acute angle formed by the line $3x + 2y = 7$ and the y-axis? Round your answer <u>to the nearest hundredth</u>.

Solution: The angle $= \tan^{-1}\left(\dfrac{2}{3}\right) \approx 33.69°$

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9. The two circles shown on the right are tangent to each other. If the area of the shaded region is 20π square units, and the sum of the perimeters of the two circles is 20π units, find the ratio $\dfrac{R}{r}$, where $R > r$.
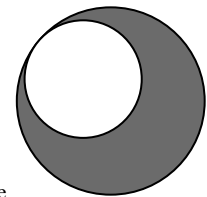

Figure not drawn to scale

Solution: $\pi R^2 - \pi r^2 = 20\pi$ and $2\pi R + 2\pi r = 20\pi \Rightarrow R^2 - r^2 = 20$ and $R + r = 10 \Rightarrow \begin{cases} R - r = 2 \\ R + r = 10 \end{cases}$

$\Rightarrow R = 6$ and $r = 4 \Rightarrow \dfrac{R}{r} = \dfrac{3}{2}$

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10. If $2x + 1$ is a factor of $3x^3 - 2kx^2 + 7x - 3$, what is the exact value of k?

Solution: $2x + 1$ is a factor means $-\dfrac{1}{2}$ is a zero of the polynomial $3x^3 - 2kx^2 + 7x - 3$. Therefore,

$3\left(-\dfrac{1}{2}\right)^3 - 2k\left(-\dfrac{1}{2}\right)^2 + 7\left(-\dfrac{1}{2}\right) - 3 = 0 \Rightarrow k = -\dfrac{55}{4} = -13\dfrac{3}{4} = -13.75.$

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11. In a tennis tournament with 8 players, each player is randomly assigned to their first-round match in the tournament bracket. Assume that the better player always wins the match. In this ideal setting, what is the probability that the second best player reaches the finals?

1st round   2nd round   Finals

Winner

Solution: The only way the second best will reach the finals is if he/she is not in the same half of the bracket as the best player. There are 4 such possibilities in the 1st round out of a possible 7 spots. Therefore the probability is $4/7 \approx .571$

12. The trough shown on the right has a rectangular base, a slanted rectangular side and right trapezoidal faces with measurements as shown.
How much water is in the trough when the water in it is 4" deep?

30"

10"

135°

10"
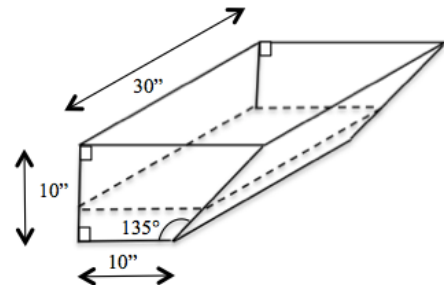
Solution: The water in the trough can be thought of a trapezoidal prism with base dimensions shown on the right.
$$\therefore V_{water} = \frac{h}{2}(base_1 + base_2) \times depth$$
$$= \frac{4}{2}(14 + 10) \times 30 = 1,440 \ in^3$$

10"   4"

4"
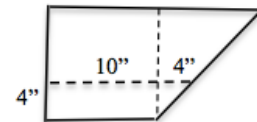
13. A function g has exactly 4 distinct zeros at –5, 2, 4 and 15.
If $h(x) = -3g\left(\frac{x}{2}\right)$, what is the sum of the zeros of h ?

Solution: The only transformation that affects the zeros is the horizontal stretch by a factor of 2, which moves the zeros to –10, 4, 8 and 30. Therefore, the zeros of h add up to 32.

14. Two trees are opposite each other across a level gorge. The tall pine tree on the right is 60 feet tall, and the angle of depression from the top of the pine tree to the top of the tree on left is 15°, and the angle of depression to the bottom of the tree is 23°. How tall is the tree on the left side of the gorge? Round your answer to <u>the nearest foot</u>.
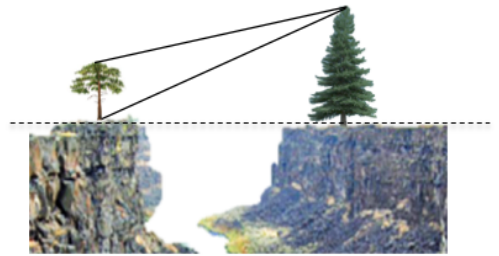

Figure not drawn to scale

Solution: Let T be the height of the tree on the left side, and let d be the distance between the two trees.

Then, $\tan 15° = \dfrac{60 - T}{d}$ and $\tan 23° = \dfrac{60}{d}$.

Substituting $\dfrac{60}{\tan 23°}$ for d and solving for T we get $T = 60 - \dfrac{60 \tan 15°}{\tan 23°} = 22.125 \approx 22\, ft$.

15. 15. In the diagram on the right, $\overline{AB}$ is a diameter of the circle and chord $\overline{CD}$ is parallel to $\overline{AB}$. $\overline{AD}$ and $\overline{BC}$ intersect in E, and $\angle DEB = 40°$. Find the ratio of the area of triangle AEB to the area of triangle CED. Round your answer to <u>the nearest tenth</u>.
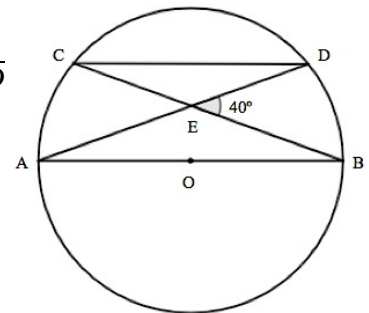
Solution: Isosceles triangles $\triangle AEB$ and $\triangle CED$ are similar, therefore, the ratio of their areas equals the square of the ratio of their corresponding sides.

Also, $\triangle ACB$ is inscribed in a semicircle $\Rightarrow \angle ACB = 90°$ and $\cos 40° = \dfrac{CE}{AE}$.

$\therefore \dfrac{\text{Area of } \triangle AEB}{\text{Area of } \triangle CED} = \left(\dfrac{AE}{CE}\right)^2 = \left(\dfrac{1}{\cos 40°}\right)^2 \approx 1.7$