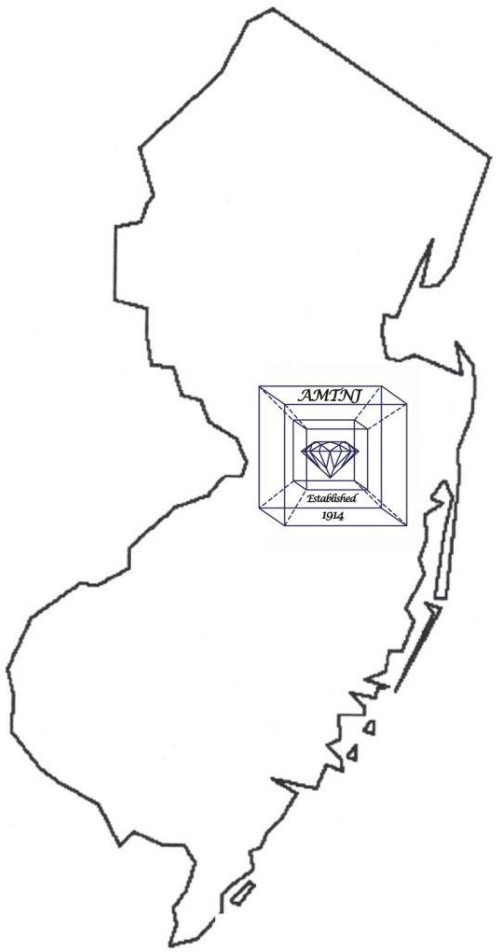

The New Jersey Mathematics Teacher

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Fall 2021

Association of Mathematics Teachers of New Jersey

The New Jersey Mathematics Teacher
Editorial Panel – Fall 2021

Guidelines for Manuscripts:

- Manuscripts should be double-spaced with one-inch margins, 12-point Times New Roman font. References should be listed at the end of the manuscript in APA format. Figures, tables, & graphs should appear embedded within the document. Do not use footnotes or head-notes.
- Manuscripts should include an abstract of 500 words or less.
- Manuscripts vary in length. The preferred length is between 1500 and 2500 words.
- Go to www.amtnj.org for more specifics regarding manuscript preparation and submission.
- All manuscripts should be submitted electronically. The electronic copy may be submitted via email attachment to Reginald Luke, Editor-in-Chief, at rkuluke@gmail.com. Author name, position, work address, telephone number, fax, and email address must appear on the cover sheet. No author identification should appear on the manuscript after the cover sheet. (The editors of the journal use a blind-review process.)

The New Jersey Mathematics Teacher
Fall 2021

Association of Mathematics Teachers of New Jersey (AMTNJ)

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EDITORIAL

Reginald Luke, Ph.D.
Editor-in-Chief
rkuluke@gmail.com

It has been a momentous year, but we are now back at work, online and in print form. In the past year so many events have transpired, politically, pandemically and climate-wise, but the creating of mathematics and mathematical education insights continues unimpeded, as this issue testifies. The mathematics community, however, has had its unfortunate losses, as we recently include Dr. Janet Caldwell, noted Rowen University mathematics professor and pre-eminent mathematics educator, among our beloved decreased. Her esteemed colleague, Dr. Jay Schiffman of our Editorial staff, has written a moving memorial in her commemoration. In this spirit of reflection, nostalgia and remembrance, Dr. Thomas Walsh, our Assistant Chief Editor, follows up with a sizeable collection of biographical sketches of notable New Jersey mathematicians in *The Mathematical Heritage of New Jersey.* The primary focus is on Bell Labs and Princeton University/Institute of Advanced Studies, but other New Jersey academic institution are represented in this installment. Next comes a bountiful cavalcade of captivating articles. Professors and Michael Finetti and Nicole Luongo of St. Peters University, Jersey City, NJ, provide us with a pandemic and technologically-relevant review of *Using Interactive Whiteboard Applications in K-12 Mathematics Classrooms*. Honor student, Cassi Zappala of Farleigh Dickinson University and mentor Dr. Nicole Hansen share with us the results of Cassi's study *The Absence of Resources for Teaching Mathematics to Students with Autism: Implications for Teachers*. Professor Rebecca Conley from St. Peters University recounts her experience with *Five Fun and Educational Activities for an Introductory Statistics Class*. Professor Lisa Rose Johnson of Rowan University and teacher Lynn Cruse suggests a design method for bilingual teachers instructing mathematics in *Implementing the Center for Applied Linguistics SIOP Interactive Activity Design Template in the Math Classroom.* Professor Aradhana Kumari from the Borough of Manhattan Community College, City University of New York, entertains us with an innovative teaching device in *A Visual Approach to Solving Equations and Inequalities involving Absolute Value*. Finally, Dr. Leonard Masse of Carteret High School shares his teaching insights about *Revisiting the Relationship between Multiplication with Fractions and Student Understanding.*

All of these articles are the products of our regional mathematics educators and students, several with interdisciplinary connections. I am sure that you too have similar insights and reflections. I encourage you to write out your thoughts, experiences and observations for submission to this publication. A good resolution for reversing the pandemic blues!

IN MEMORIAM: DR. JANET H. CALDWELL

Jay L. Schiffman, Associate Editor - New Jersey Mathematics Teacher

It is with a heavy heart indeed that I write this small tribute to my dear friend and esteemed colleague, Dr. Janet H. Caldwell, long time Professor of Mathematics at Rowan University (1983-2016) in Glassboro, NJ, who we lost far too soon on March 13, 2021. I will attempt, albeit unsatisfactorily, to highlight Janet's numerous accomplishments and contributions to our mathematics education community. At best, my reflections are incomplete; for a complete accounting of Janet's achievements and how she touched so many lives in both the education and mathematics communities would be next to impossible. On the other hand, as a very humble person and scholar, Janet never desired to have her accomplishments publicized.

The improvement of mathematics education was always Janet's desire. One of the most empowering programs in New Jersey, which in no small measure helped to achieve this goal, was McSiip (Mathematics and Computer Science Instructional Improvement Program) initiated at the then Glassboro State College. Through numerous grants (Janet was a prodigious grant writer, which generated millions of dollars during her esteemed career to aid in the process), numerous workshops for teachers in mathematics, science and computer science were created. In our numerous discussions on mathematics education throughout our many years as colleagues, I always knew that Janet was frustrated with the dichotomy of being able to carry out mathematical procedures versus having a deep understanding of mathematical content which is paramount to successful teaching. In The 15th "Good Ideas in Teaching Precalculus And …" in March 2001, our mutual friend of many years, Professor Joe Rosenstein of Rutgers University, who has expertly managed the conference for more years than we may care to remember anointed Janet to present our Plenary Session. The title "Math > Gimmicks + Rules" was such a poignant and timely presentation. As a mathematics person not trained in mathematics education, but who has gravitated towards it over the past two decades (in no small measure due to Janet, Joe, Eric and three other colleagues in mathematics education at Rowan: Bill Smith, Marlena Herman and Karen Heinz), I have witnessed a new generation of thinking. I remember a most timely point articulated by Eric when we co-authored an article for our Journal in 1999 entitled "A Most Unusual Social Security Number": Jay: You have to think of this situation as a mathematics educator, not as a mathematician. Needless to say, Eric's point resonated with me!

Another fond moment of working with Janet on our hiring committee entailed the following: During our interview with candidates, we required them to present a sample lesson. For someone teaching a calculus class, the topic centered on developing the exponential function. A question posed in this setting was "Why is any non-zero base raised to the zero power equal to one? " The typical response we received from candidates with advanced degrees was "It is a rule." Sorry, that did not cut it for Janet or any one of us. We were seeking pattern-based reasoning. I have found over the past twenty years that while even some of the best mathematicians can solve problems with aplomb, much of their work is often highly procedural in nature and filled with CODE WORDS such as "cancel" and "plug-in" (which is not a mathematical term) among others translating to student learning falling far short of where it should be.

Janet was an incredibly passionate and caring person as well as a luminary in the field of mathematics and mathematics education. She was always open to sharing her gift with the general populace. As we celebrate her amazing career and a life well lived, I fondly remember working with her on the assessment committee for the department. In addition, we worked closely during her tenure as the founding president of our sister organization The New Jersey Association of Mathematics Teacher Educators (NJAMTE). This organization was founded in October 2006 at The NCTM Eastern Regional Conference and Exposition jointly co-hosted with AMTNJ and convened at The Atlantic City Convention Center in Atlantic City, NJ. I was honored to be asked to serve as the four-year college representative for the organization during Janet's presidency and later as secretary during the presidency of Cathy Liebars of The College of New Jersey (another superb mathematics educator who I feel privileged to call a friend) as well as the liaison for AMTNJ. Janet was a joy to work with and I still serve the organization as the liaison to AMTNJ. I had the good fortune of viewing numerous presentations delivered by Janet over the years. In August 2018, I viewed a presentation offered by Janet hosted by our sister organization PCTM in Harrisburg, PA. Alas it was the last time I was able to see my dear friend in action. In short, Janet was Janet, and her middle school presentation was artful and highly informative. One of the most redeeming qualities possessed by Janet that I have always admired was her sense of selflessness. It was never about her. It was always focused on the lives she touched, and anyone who had the pleasure of working with her immediately viewed this through that lens. At every presentation Janet delivered, a sense of community among the participants radiated, and Janet's signature was that "we are all friends here".

Janet was the recipient of numerous awards including "The Max Sobel Award" from our organization and "The Ernest Boyer Award" from The New Jersey Section of The Mathematical Association of America in 2000 for excellence in teaching as well as "New Jersey's Teacher of The Year" in 1994. Janet was President of AMTNJ during the 1990-91 academic year as well as a major player in the NCSM (National Council of Supervisors of Mathematics) and the NCTM. Being a true leader and a scholar encompassed only a small sample of Janet's attributes. She will be sorely missed, but please remember that when we achieve and help others in our field, Janet's spirit will be smiling down on us. If there is an irreplaceable individual in the field of mathematics education, it is Janet and I feel both blessed and richer for having known her as both a colleague and dear friend. As I plan to transition to retirement (as a number of you may already know) at the conclusion of this academic year, I can only hope that I was able to have contributed even one percent of what Janet accomplished. My best wishes for better days ahead for all. I will stop here and now present some testimonials to Janet’s legacy from two sources that will expand on her numerous and stellar contributions: Our AMTNJ organization’s tribute written by Neil and Stephanie Cooperman as well as a small tribute from my friend and esteemed colleague Dr. Eric Milou.

FROM AMTNJ ON JANET'S RETIREMENT:

Janet Caldwell graduated from Rice University with her Bachelor’s Degree in Mathematics and French, *C’est vrai*? Janet taught mathematics and computer science for a year in Spring, Texas before matriculating at the University of Pennsylvania. She earned her Master of Arts Degree in Mathematics Education and she continued on to earn her Ph. D in Mathematics Education Research with a minor in Evaluation, garnering the Phi Delta Kappa Outstanding Dissertation Award and the William B. Castetter Distinguished Service Award in the process.

While Janet was earning her advanced degrees, she also served as a lecturer and research assistant at The University of Pennsylvania; she was a lecturer at Cabrini College in Radnor Pennsylvania and a teacher at Lansdowne-Aldan High School in Lansdowne, Pennsylvania, all, more or less simultaneously. She worked for Research for Better Schools for five years before accepting a position at Rowan University.

Janet (Caldwell), along with Warren Crown and Joe Rosenstein, were the primary authors of the “New Jersey Mathematics Curriculum Frameworks” in 1996. Stephanie and I each had the honor of serving on one of the sub-committees. These “Frameworks” were quickly adopted as the New Jersey Core Curriculum Content Standards. She helped to oversee the revisions for the next fifteen years.

Janet was a founding member of the New Jersey Association of Mathematics Teacher Educators and served as that organizations’ president from 2007-2010. Janet was also a founding member and has served on the Board of Governors of the New Jersey Mathematics Coalition from 1992 until the present. She was a founding member of the New Jersey Association of Women in Mathematics, and she was a member and President of the New Jersey Teachers of Elementary Mathematics.

Janet has served both the National Council of Supervisors of Mathematics and the National Council of Teachers of Mathematics in more capacities than I have the time to recount tonight. Janet has been and continues to be a frequent speaker throughout the region and the country. She is always one of the most popular and sought-after presenters at the Association of Mathematics Teachers of New Jersey’s yearly conferences. Janet has been an engaging, hands-on presenter, and we only wish that more teachers would take her recommendations to heart.

Janet was President of AMTNJ in 1990-91. While serving AMTNJ, she was instrumental in reorganizing the structure of the association, creating the position of Administrative Assistant/Office Manager to oversee the daily minutiae of running the state's oldest and largest organization devoted to mathematics education. In addition, she oversaw the restructuring of the association to ensure regional representation with appropriate staggered terms of service and she re-organized the succession of The Ladder, the executive leadership team of the association. Janet was the Editor of the AMTNJ Monograph, "The AMTNJ Calculator Handbook". Janet was awarded AMTNJ's most prestigious award, "The Max Sobel Award for Outstanding Service and Leadership" in 1994.

Many AMTNJ Presidents, having survived the torture of leading New Jersey's 102-year-old organization, have faded into history as their time of service ended. Not Janet. She has remained as a stalwart and active participant, attending and speaking at most of our conferences, presenting at least once a year via our WebEx presentations, and you can depend upon Janet to be in attendance and voicing her opinions at our Annual Business Meeting each year. Janet has always remained a presence, dispensing her words of wisdom in a quiet, strong way to help a new speaker, to help AMTNJ, and always to offer sage advice.

Janet has always been the strongest advocate and a true champion for South Jersey. She has served as a mentor for many educators. In particular, Eric Milou comes to mind. Janet is known for being unable to lead a meeting without her omnipresent bottle or can of DIET COKE. Bob Riehs recently made me aware that Janet is a fantastic cook and has one of the most impressive home kitchens that he has ever seen. Stephanie and I are looking forward to an invitation from Janet.

Janet's retirement is a terrible loss for Rowan University and mathematics education in New Jersey. We can only hope that she will prolong her service to the greater mathematics community in some capacity, and that she will continue to be a steadfast player to the benefit of AMTNJ in the future.

On behalf of President Kristie Prokop and the Association of Mathematics Teachers of New Jersey, I would like to award this plaque to Dr. Janet Caldwell. It reads:

The Association of Mathematics Teachers of New Jersey
Gratefully Acknowledges
Janet H. Caldwell for
Many Years of Service to the Professional Mathematics Community
and for Many Years of Service to AMTNJ

Congratulations on your retirement!
May, 2016
Once again, congratulations on your retirement!

FROM DR. ERIC MILOU, ROWAN UNIVERSITY PROFESSOR OF MATHEMATICS:

Regrettably, I have been asked to inform the department that our former colleague, Janet Caldwell, passed away this past Saturday (March 13, 2021). Her husband informed me her family was by her side and able to communicate with her from time to time and hold her hand at the end. The family wishes no calls and has no plans for a memorial at the moment.

Janet was a member of our department for over 30 years and retired in 2016. Her accomplishments are too many to mention and she would never want me to list them as that was not what Janet stood for. Instead, I will tell that in the 19 years that we worked together at Rowan, there was no one who enriched my life more and those of countless Rowan students and math educators throughout the country. Janet's reputation was known from coast to coast and is the main reason I came to Rowan in 1997. Her students would tell you that she brought mathematics to life, highlighting its beauty and elegance. And as a mutual friend once said of Janet, "Her eyes would sometimes actually twinkle when she caught a student in an aha moment." She was an author, gifted teacher, mentor to many, proud mother, grandmother, loving wife to husband John, and most importantly a great friend. She leaves a legacy that will live on in the hearts and minds of all who were privileged enough to have known her and those lucky enough to be taught by her. Rest in peace, my dear friend Janet.

To summarize, Janet was one of a kind. A final fond memory of Janet's retirement celebration in May 2016 was the grace of her beautiful family. Her fabulous husband John who in his own right is extremely accomplished as an eminent patent attorney (recently retired after many successful years) in Center City, Philadelphia greeted both Eric and myself at the conclusion of the ceremony and articulated the following: "Thank you for being a friend to Janet." That was such a nice touch on John's part. On the other hand, all of us in the mathematics education community should be saying to Janet whose spirit I know is smiling down on us: "Thank you Janet for being a friend to all of us!" To make certain that Janet's legacy endures, Eric has created the Janet Caldwell memorial scholarship and contributions to the Janet Caldwell Memorial Scholarship Fund can be made at: https://go.rowan.edu/caldwellscholarship.

The Mathematical Heritage of New Jersey

Thomas P. Walsh, Ed.D.
Kean University
tpwalsh@kean.edu

The state of New Jersey has a rich history of mathematical discoveries. There are many institutions of higher education in New Jersey, and there are many mathematicians who have worked at these institutions, colleges and universities over the years. This paper will focus on two preeminent institutions that are New Jersey based: Princeton University, and the research and engineering institution of American Telephone and Telegraph: Bell Labs. As well, mathematicians and mathematics educators from other institutions, will be mentioned.

This is far from a comprehensive review of all the mathematical contributions (and mathematicians) in the Garden State, but it attempts to highlight the major contributors to the field. It is not my intention to miss anyone of note, however, and I welcome any comments, or suggestions. I can be reached at: tpwalsh@kean.edu.

AT&T Bell Labs, Murray Hill, NJ (now known as Nokia Bell Labs)

AT&T Bell Labs, Holmdel, NJ (now known as Nokia Bell Works)

Harold D. Arnold joined A. T. & T. in 1914 (before Bell Labs). He helped to perfect the audion, a glass tube that eventually became known as the vacuum tube. The vacuum tube was instrumental in amplifying telephone signals allowing them to be transmitted over long distances. Vacuum tubes were instrumental in many electronic devices, such as the radio, television, and radar. He also demonstrated that a radio receiver could reproduce a voice signal, paving the way for AM radio, perfecting this transmission system that had been introduced by Alexander Graham Bell in 1880. As well, it was Arnold who showed experimentally that electrical theory could be used to improve the sound of phonographs.

Vic Benes (Image not available) came to Bell Labs in 1953. He developed the mathematics for optimizing telephone switching. That is, routing a call to its intended party, with as little use of equipment as can be managed. As calls come in to a switching machine, the challenge is to get it routed to their appropriate end points. Vic Benes developed the mathematics to make those routings happen. He envisioned the switching machine as a finite-state automation, and used queueing theory, combinatorial mathematics, and network theory, developing a uniform framework to solve the switching problem. In his book, he described the statistical problems of traffic congestion and the problems of comparing networks with respect to traffic capacity.

Robert Calderbank joined Bell Labs in 1980, eventually becoming vice-president for research there. He coinvented (along with Vahid Tarokh and Nambi Seshadri) space-time codes, which are a fundamental technology in the development of wireless communications. As well, he helped in the management of “big data’; the manipulation of very large sets of data to find patterns in interactions among people.

John R. Carson (Image not available) joined A. T. & T. in 1914 (before Bell Labs). He experimented with radio telephony, and in 1915 he invented single sidebanded modulation, that allows for multiple telephone calls on a single telephone line. In 1922, he developed the mathematical underpinnings of frequency modulation (FM). In this 1922 paper, he introduced the Carson Bandwidth Rule, that defines the approximate bandwidth requirements of communication system components for a carrier signal that is frequency modulated by a continuous broad spectrum of frequencies instead of a single frequency.

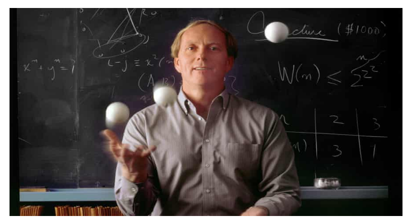
Fan Chung came to Bell Labs in 1974, and worked there till 1994. Since 1994 she has taught in several major universities, and presently she is at the University of California, San Diego. She has worked in the field of combinatorics all of her career, specifically in the area of Ramsey theory. She has produced a great number of papers (over 200) in graph labeling, graph decompositions, graph algorithms, and extremal graphs, as well as spectral graph theory, parallel structures, and applications of graph theory in internet computing.

Ingrid Daubechies joined Bell Labs in 1987. She developed compactly supported continuous wavelets using quadrature mirror filters. Previous to this, Fourier analysis was used. Dr. Daubechies' method allowed for wavelets to be processed digitally, and thus greatly enhanced cell-phone and computer processing. She put her research out on orthonormal bases of compactly supported wavelets while at Bell Labs. One of the best-known applications of this is the FBI's use of wavelet transformation to digitize fingerprint records. Dr. Daubechies taught at Rutgers and Princeton during her time in New Jersey.

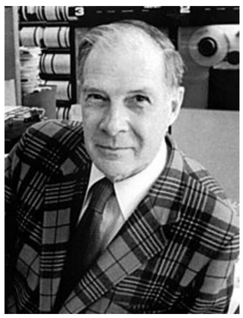
Ronald M. Foster studied at Harvard University, earning his degree in mathematics in 1917, the same year he joined A. T. & T. (later Bell Labs). Foster worked on network analysis, which is the description of a network in mathematical terms. Going the other way, network synthesis is the development of a network that obeys some mathematical description. In 1924 Foster developed a theory that prescribes, in mathematical terms, the reactance as a function of the frequency of a network. It is called the Foster Reactance Theorem. Reactance is the opposition of an element in an electrical circuit to any change in an electric current or voltage. In 1943, Foster left Bell Labs to become head of the Mathematics Department at the Polytechnic Institute of Brooklyn.

Thornton Fry started the mathematics section (as it was then called) in 1922, with himself and a small staff of assistants to help in the computations. It was initially formed to serve the many engineering sections in the company, but by the 1930s the mathematics department was a separate section employing many mathematicians, and the term "industrial mathematics" was used to describe the work these people did for other areas in the Bell System. Since switching calls in a network becomes a probability problem, Fry's early work in switching telephone calls led to his deep development of probability, culminating in his publishing a probability textbook in 1928. It was his mathematics department that later developed the digital computer. Using Cauchy's principle, he invented a machine, the isograph, a mechanical analog computer (along the lines of Boole) that calculated the roots of polynomials.

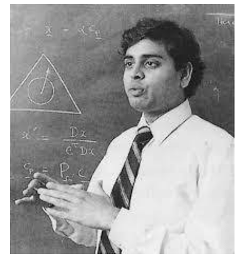
Ed Gibert came to Bell Labs in 1948, after earning his Ph.D. in Physics at M.I.T. He worked there till his retirement in 1996. His work was quite varied. He, along with Pollack and Hwang developed a minimal length tree (when extra vertices are allowed). Another area of his interest was in coding theory. He found that, when coding a program, thinking of the code as spheres packed into a space can reduce transmission, and therefore conserve transmission power. He also developed Gilbert tessellations; a theory of crack formation whereby cracks form in a random fashion, and spread until they run into another crack.



Ronald Graham joined Bell Labs in 1961 and worked there until 1999. Graham is a trampolinist, an acrobat, and an accomplished juggler. He is in the Guinness Book of World Records for using the highest number ever in a mathematical proof, known as "Graham's Number". At Bell Labs, he worked on worst-case scheduling theory, and did pioneering work in the field of computational geometry. Multiprocessor computers are computers that have many processors, and can tackle a problem by dividing it into subtasks. In some problems, though, one task has to be completed before another task can begin. It is ideal to solve a problem with the least number of processors, and the smallest amount of time. Graham explored these issues, and found, somewhat counterintuitively, that sometimes an algorithm can complete all the tasks and come up with a solution feaster if the task length is increased. He also did the first work on Ramsey theory, which is a branch of discrete mathematics involved with a large generalization of the pigeonhole principle. Graham pointed this generalization out, and showed the theorem's applications to many other areas of mathematics.

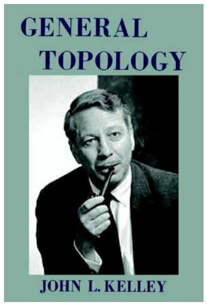


Richard Hamming joined Bell Labs in 1946 after working on the Manhattan Project. He along with Claude Shannon, Don Ling, and Brockway McMillan, called themselves the "Young Turks" (all were around 30, they shared a common bond in scientific research, and all were somewhat unconventional). Hamming was a computer evangelist, and often helped other mathematicians solve large computing problems using the computer, when their desk calculators could not. Errors would come up in computers, and it was hard to detect them. Hamming came up with a way for the computer to self-correct. By adding a few bits to a block of data, the computer could find an error in code, and the place where the error was, as well. He worked on more and more efficient ways of error correcting, these techniques eventually became known as Hamming Codes, and have been used ever since in telephone switching systems.



Narendra Karmarkar came to Bell Labs in 1983, and worked there until 1997. He also worked at the Institute for Advanced Studies in Princeton for a year (1997 – 1998). He came up with a polynomial algorithm that solves linear programming problems, known as the interior point method, also known as the Karmarkar Algorithm.

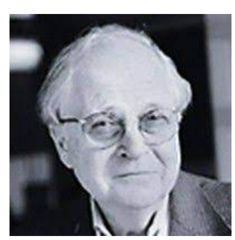
Mervin Kelly worked at Bell Labs from 1925 until 1959. As World War II was coming to a close, Bell Labs committed to developing solid-state electronics, to replace vacuum tubes in electronic devices. This change in internal components offered the promise of miniaturizing devices. He was convinced that basic research in semiconductors, conductors, dielectrics, insulators, capacitors, and resistors, would result in smaller, cheaper, and more durable electronic devices. The first big breakthrough came in December, 1947, with the invention of the transistor, allowing a small electrical current to modify (amplify or attenuate) a much larger current.



John L. Kelly came to Bell Labs after a time in the oil fields of Texas (in the mid-fifties). He was heavily involved with computers and programming, developing an easy-to-use programming method. The compiler system he (with C. Lochbaum and V. Vyssotsky) created was the Block Diagram Compiler (BLODI). He also created (with Lochbaum and L. Gerstman) a voice synthesizer on the IBM 704 being used at Bell Labs at the time. They recorded the song *Daisy Bell*, which impressed Arthur C. Clarke so much that he put it in his novel and in one of the iconic scenes in the movie *2001: A Space Odyssey*. He also, along with Claude Shannon, developed a theory for playing odds in games of chance, and this game theory is still used in investing today.

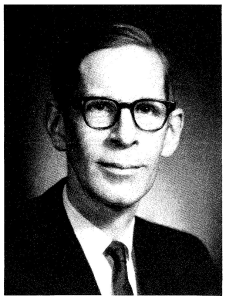
Joseph Kruskal came to Bell Labs in 1959, and worked there until 1993, but continued to conduct research at that institution. He found an efficient method for the minimum spanning tree. That is, a network of links connected by a minimum number of wires. This became the basis for billing at AT&T. He also developed mathematical and computer methods for improved computation of multidimensional maps, or multidimensional scaling (MDS).

Jeff Lagarias worked at Bell Labs from 1975 till 2002, when he moved to the University of Michigan. Although his original work was in analytic algebraic number theory, much of his recent work has been in theoretical computer science. He has worked in many fields of mathematics, however, and considers himself a mathematical generalist. At Bell Labs he wrote a paper on the history of the 3x + 1 Conjecture, also known as the Collatz Problem, the Syracuse Problem, Kakutani's Problem, Ulam's Problem and Hasse's Algorithm. It states that, given any positive value x, repeated iterations of 3x + 1 will tend toward 1.



Henry J. Landau was a mathematician who joined Bell Labs in the late 1950s.

He developed the mathematics that eventually allows engineers and scientists to calculate the limits in analog to digital conversion. He also did computer simulations on transform coefficients to improve picture quality.



Donald Ling was a mathematician who joined Bell Labs in the late 1945, coming from the National Research Defense Committee, and was heavily invested in the Nike systems of antiaircraft and antimissile systems, along with John Tukey. He was a respected and much sought-after member of committees discussing national defense. As such, much of his contributions to his country is classified. He wrote a book (published in 1908) that brings technical mathematics down to the nonmathematician. He, also, was one of the “Young Turks”.

Brockway McMillan joined Bell Labs in 1946, left there to work for the US Air Force from 1961 to 1965, and rejoined Bell Labs in 1965, retiring in 1976. He, along with L. Kraft, developed the Kraft-McMillan Inequality, which gives necessary and sufficient conditions for the existence of prefix code, or a uniquely decodable code, in coding theory. He was one of the “Young Turks”.

F. Jessie (Collinson) MacWilliams joined Bell Labs in 1958 as a computer programmer. She got interested in coding theory after attending a talk by R. Bose. Her Ph.D. thesis contains arguably, one of the most important theorems in coding theory. The equations on coding theory she wrote describe how the weight distribution of a linear code compares to the weight distribution of its dual code. When they are equal, the code is called self-dual. The equations she developed are widely used by coding theorists to obtain information about self-correcting code, and help in combinatorial designs. She identified a set of functions (now called the MacWilliams Identities) that any linear code must satisfy. She wrote a book with Neil Slone: *The Theory of Error-Correcting Codes*, which is an enormous tome containing 1500 references, and covering many, many areas of coding theory. As well, she worked on cyclic codes, generalizing them to abelian group codes.

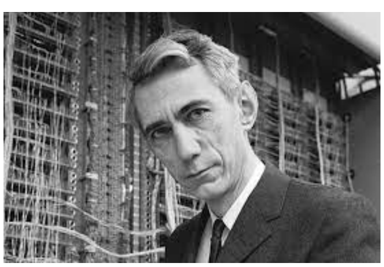
Edward L. Norton Started at Western Electric in 1922, before it became Bell Labs. He was interested in many areas of research, including network theory, acoustical systems, electronics apparatus, and data transmission. This last area led to his development of the Circuit that bears his name: the Norton Equivalent Circuit. Based on Norton's Theorem, it shows that a large, complex electrical circuit can be expressed, with a bit of algebra on electrical equations, to be a simple circuit with a single voltage, resistance, and current.

Andrew Odlyzko worked at Bell Labs from 1975 to 2000, and is presently at the University of Minnesota. He has over 150 papers to his name, and worked in the fields of computational complexity, cryptography, number theory, combinatorics, coding theory, analysis, and probability theory. In 1995 he circulated a paper within Bell Labs predicting that large research labs (such as Bell Labs) were on the decline due to a complex of several forces, and not simply due to narrow-minded management looking to maximize their profits.

Henry Pollak joined Bell Labs in 1951, and was named director of the mathematics department there in the early 1960s. He has done much work on discrete mathematics, network theory, function and probability theory. He is a pioneer in the applications and modeling in mathematics education. His belief is that students in all grades (kindergarten through college level) can and should be taught the applications of mathematics and how we model with mathematics. It is his belief that teaching this allows students to understand the vital link between mathematics and our modern world. He was one of the original members of the School Mathematics Study Group (SMSG). He is the author of close to forty papers on these and other topics.

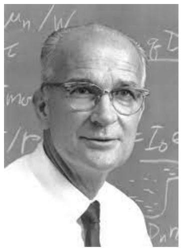
Robert Prim came to Bell Labs after World War II. His major contribution was in the area of graphs, specifically minimum spanning trees. The problem, for the phone company, is to get a call from one person (node) to another covering a minimal amount of distance. Fry developed an elegant and very simple algorithm (an algorithm that bears his name) for minimally spanning a tree, one that is especially adaptable to computers. He published a book in 1957: *Shortest Connection Networks and some Generalizations*, on the spanning of trees.

John Riordan (Image not available) came to Bell Labs in 1926, and worked there until 1968. He worked extensively in combinatorial mathematics, and wrote the classic book on combinatorics: *Introduction to Combinatorial Analysis*. He authored many papers and books on combinatorial and queueing theory. As well, he was a prolific poet and edited the literary magazines *Salient* and *The Figure in the Carpet*, both published by the New School for Social Research.

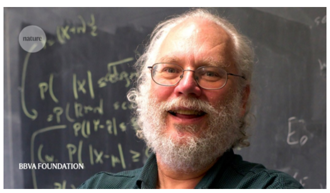


Claude Shannon was a mathematician who worked for both the Institute for Advanced Study (at Princeton University) as well as Bell Labs. It was Dr. Shannon who developed the system by which information could be encoded using a series of ones and zeros, thus allowing Boolean Algebra to be utilized to analyze logic circuits. This allowed information to be transmitted within a computer, and across telephone lines, and thus enabled telecommunications and the modern computer age. A second discovery he made was the notion that a source of information transmits according to probability law. In 1948, his paper: "A Mathematical Theory of Communication" formed the mathematical foundation of electronic communications. It was groundbreaking in its scope, and an entire field was born. The central idea is that a message can be treated as a single, physical entity, like an object. He coined the name "Information Theory" to describe the study of signaling systems from general ideas to very specific communications systems. It is for this reason that he is considered the father of the information age. He was the fourth member of the "Young Turks".

Paul Seymour is a mathematician who joined Bell Labs in 1983, and in 1996 he became a professor at Princeton University. His major field is combinatorics, graph theory, and optimization. He has done ground-breaking work on regular matroids, totally unimodular matrices, and on the four-color theorem, among other contributions to mathematics.

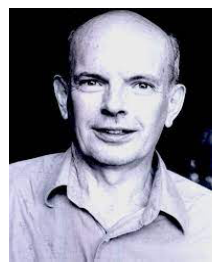


William Shockley worked at Bell Labs from 1936 to 1955, and was the inventor, along with W. Brittan, and J. Bardeen, of the transistor, which eliminated the need for vacuum tubes.



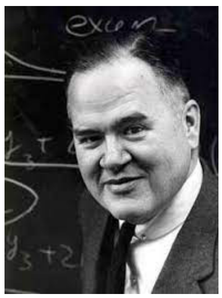
Peter Shor came to Bell Labs in 1986, and worked there until 2003. He is the author of an algorithm called "The Factoring Algorithm" by him, and "Shor's Algorithm" by everyone else. It is a quantum algorithm, that is, it is an algorithm that runs on a quantum computer. It calculates all prime factors of a number, N. This is very important when we have extremely large prime numbers that are used to encrypt data, such as bank accounts and medical records, among other things.

David Slepian was hired at Bell Labs in 1950, and worked there for the rest of his career. His early work at Bell Labs was in the area of detection theory, and this led to his interest in prolate spheroidal wave functions. He collaborated with Pollack and Landau on the characterization of the bandwidth of electrical signals. He discovered the possibility of singular detection, in detection theory. However, his most important contribution to mathematics was in the area of coding theory. He introduced the idea of a binary group code. He introduced the notion of expressing the coding problem in a contemporary algebraic setting, and to analyze codes algebraically.



Neil Sloan has worked at Bell Labs from 1968 to 2012. His major works include contributions to combinatorics, error-correcting codes, and sphere packing. He wrote a book on sphere packing with John Conway. He is perhaps best known for his creation of the Online Encyclopedia of Integer Sequences (OEIS), which he serves as president.

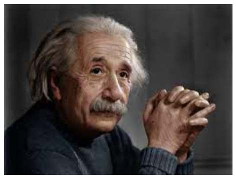
George Stibitz worked at Bell Labs from 1930 to 1964. In 1937 he invented a general-purpose relay computer, which made simple arithmetic calculations in binary. The next year he modified this computer to handle arithmetic operations on complex numbers. This computer, known as the Model 1, was used continuously from 1940 to 1949. It was the first general purpose digital computer to be used, and so Stibitz is considered to be the father of the modern digital computer.



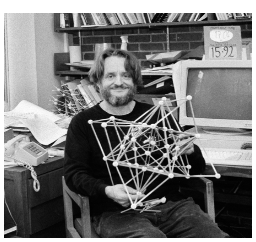
John Tukey started at Bell Labs in 1945, but spent most of his time in Princeton University. He contributed a great deal to many military projects, principally the Nike anti-aircraft missile project. His major contributions to mathematics were in the area of statistical research. Along with J. Cooley, he developed the far-reaching Fast-Fourier Transform, an ingenious and highly efficient method of regrouping the computations of Fourier coefficients used in signal processing, among other areas of mathematics. It made calculations in many areas of engineering much quicker to perform. He also developed, along with M. Bartlett, the modern digital methods of spectrum estimation.

Peter Winkler was the last director of the mathematics department of Bell Labs, and later Lucent Technologies. He has been a professor at several colleges, and presently he is a professor of mathematics at Dartmouth University. He has worked in the areas of cryptology, discrete mathematics computation and probability theory, and marine navigation. He is the author of 125 research papers and 8 patents, and the author of two books on puzzles

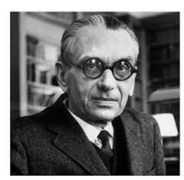
Institute for Advanced Study, and Princeton University



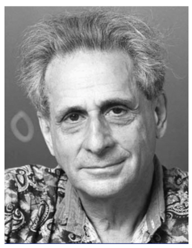
Albert Einstein, often called the greatest mind of our age, was a physicist, of course, but he proved one of the more iconic mathematics equations: $E = mc^2$. This shows that mass can be converted into energy, and thus it presages the atomic age. The Institute for Advanced Study at Princeton was set up specifically to invite Albert Einstein to America. Some people say that Einstein was not terribly good at mathematics, and it is true that he sometimes needed help with some of his calculations, but once he set his mind to a concept, he learned it well. With his Theory of Special Relativity, and his Theory of General Relativity, he remade Physics and our view of the cosmos. He showed that time slows down closer to the speed of light, and space bends around massive objects, such as black holes and neutron stars. It was his thought experiments (and his great imagination) that allowed him to develop these world-changing concepts in our understanding of the universe.



John Conway was an amazingly versatile mathematician who contributed much in the fields of number theory, Algebra, Geometric topology, combinatorial game theory, Topology, and Group Theory. He wrote three volumes on the mathematical basis behind many, many different games. He was constantly engaging mathematicians, students and others in mathematical questions, and carried pennies, cards, dice, and other props in his pockets to use in engaging people he met in mathematical games. One of his most famous games (featured by Martin Gardner in *Scientific American*) is The Game of Life. It is not the board game published by Milton Bradley, but rather a game played on an infinite grid, that allows for birth, death and continued life of an organism based upon a small set (three) of simple rules. The achievement he is most proud of was his invention of the Surreal numbers, which is a continuum of numbers including not only the Real Numbers (Integers, fractions, irrational numbers and transcendental numbers), but infinitesimal and infinite numbers, as well.

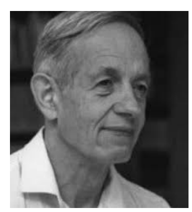


Kurt F. Godel worked at the Institute for Advanced Study from 1940 until 1953. He was a great friend of Albert Einstein. He is generally recognized as laying the foundations of mathematical logic. He is, perhaps, most well-known for his theory of incompleteness. This theory states that, in any system (or branch) of mathematics, no matter how many propositions are proven, there will always be some propositions that cannot be proven within that system. One will need to go outside the system, thus creating a larger system, with its own unproven propositions. The implication is that all systems are, by definition, incomplete, and that each and every system contains more true statements that can be proved by its own defining set of rules.



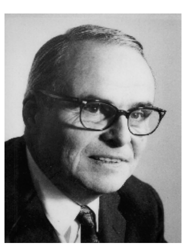
Martin Kruskal started in Princeton University on Project Matterhorn, which eventually became the Princeton Plasma Physics Laboratory. It was here that he contributed perhaps the greatest utility: that of showing that nuclear energy (particularly nuclear fusion) could be harnessed in electrical generating plants. He did much of his early work in plasma physics, writing many papers in how to
utilize plasma in areas such as how to achieve a practical fusion reaction. The key is containing it through hydromagnetic(plasma) stability. He did much research in this a
rea, publishing the first calculation of plasma stability. He originally worked as a professor of Astronomy, but gradually moved into the area of applied mathematics, being appointed professor of mathematics in 1979, the same year he retired from Princeton. That same year he accepted a position at Rutgers University, being named the David Hilbert Chair of Mathematics at Rutgers. At Rutgers, he worked on soliton equations, asymptotic analysis, non-linear differential equations, and surreal numbers.

Deane Montgomery worked on and off at the Institute for Advanced Studies from 1934 to 1992, ending there as an emeritus professor for 12 years. His main work was in topology, and he was one of the contributors of Hilbert's fifth problem. He collaborated with C.T. Yang on differential topology through the study of group theory on homotopy 7-spheres. He was very self-effacing, and mentored many younger mathematicians, encouraging them and suggesting areas of research to them.



John F. Nash is a Princeton Professor of Mathematics, and author of a number of papers on game theory. His ideas have provided a great deal of understanding as to how chance and other various forces influence complex systems in everyday life. He has developed theories that have influenced market economics, accounting, and military theory, among other areas. He has also worked in differential geometry and partial differential equations. The movie "A Beautiful Mind", starring Russell Crowe, is loosely based on John Nash's life.

John Von Neumann is considered to be one of the greatest mathematicians of the 20th century. He was doing 8-digit division in his head by age 6, he was doing graduate level mathematics at age 12, and he earned a Ph.D. in mathematics and chemical engineering at age 23 (in 1926). His first book was published in 1932 in the very new field of quantum mechanics. He came to Princeton two years before in 1930, and in 1931 he became a professor there. In 1933, the Institute for Advanced Study was formed, and John Von Neumann was one of the six named to full-time staff in the School of Mathematics, along with Albert Einstein. He worked out in Los Alamos on the atomic bomb, solving the problem of how to ignite an uncontrollable nuclear reaction (implosion). He published papers in the areas of set theory, functional analysis, geometry, and numerical analysis.



Al Tucker came to Princeton in 1929 as an instructor, and he earned his Ph.D. in 1932 there. He was promoted through Assistant, Associate, and finally to full professor at Princeton by 1946. He was chairman of the Princeton Mathematics Department from 1953 till his retirement in 1974, mentoring many fine mathematicians, including John Nash, Marvin Minsky, Torrence Parsons, and Lloyd Shapley. During World War II he worked on the Princeton Fire Control Project (using applied mathematics for directing naval and Army gunfire controlled by directing and training). He developed the mathematical technique of linear programming, to optimize the use of scarce resources. As well, he made important contributions to topology, and he developed much of gaming theory. He posited the famous "Prisoner's Dilemma" problem.

Hassler Whitney worked as a mathematician at the Institute for Advanced Study from 1952 – 1977, and then as an emeritus professor from 1977 – 1989. Active in geometry, topology, and graph theory, Dr. Whitney was also an avid mountaineer, climbing heights in New England as well as mountains in the Swiss Alps. He also was an amateur musician, playing both the violin and viola. He was one of the founders of singularity theory. He worked on the four color problem, and the proof of that problem by computer is based largely on his earlier groundwork. His main contribution to mathematics, however, is in topology, and particularly in the area of manifolds. He developed ground-breaking work in differential and integral manifolds, and wrote a textbook on differential manifolds that is still used today. His work in geometry encompassed differential and integral geometry. He was also interested in elementary mathematics, and once spent four months teaching pre-algebra to seventh graders. He worked to reduce math anxiety among elementary pupils.

Andrew Wiles is a mathematician on the faculty of Princeton University, where he teaches both undergraduate and graduate mathematics classes. He feels it's important to teach both levels of mathematics to students. His specialty is in the area of number theory. He is most famous for proving Fermat's Last Theorem (with help from Dr. Richard Taylor of Harvard University) in 1995. This theorem states that there are no solutions to the equation $a^n + b^n = c^n$, where $n>2$. He worked on it secretly for seven years before finding the proof.

Noted New Jersey Mathematics Educators

Howard F. Fehr (image not available) was a professor of Mathematics Education at Columbia University's Teachers College from 1948 to 1967. He was author of many books on mathematics, but is best known for being the founder of the New Math, in the early 1960's. His report: "New Thinking in School Mathematics", issued in 1961, made the case for the new mathematics by explaining that the traditional methods of teaching mathematics (memorization of tables and procedures without understanding of them) would not properly prepare citizens of the new, technological society being shaped. He directed the Secondary School Mathematics Curriculum Study, from 1965 – 1973. This study was instrumental in transforming mathematics education.

Gerald Goldin is a professor of Mathematics, Physics, and Education at Rutgers University. He has authored 180 scholarly articles, supervised 12 doctoral dissertations, and has authored several books on mathematics education. He was the first director of the Rutgers Center for Mathematics, Science, and Computer Education, and he organized the Statewide Systemic Initiative in

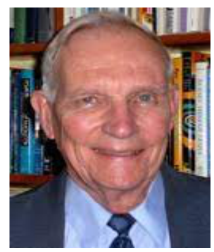
Mathematics, Science and Technology Education, funded by the NSF, and presently he is the Principal Investigator in the MetroMath project, which partners with three universities to improve mathematics education in low-income inner-city neighborhoods.

Daniel Gorenstein worked at both the Institute of Advanced Study, Princeton and Rutgers University. At a very early age (12) he taught himself Calculus. He went to Harvard university, where he became very interested in finite groups. During World War II, he taught mathematics to soldiers. After the War, he continued his studies, earning his Ph.D. at Harvard. In his thesis in algebraic geometry, he introduced Gorenstein Rings (which are commutative, Noetherian local rings). He worked at Clark, Cornell, and Northeastern Universities before moving to New Jersey. He was a member of the Institute of Advanced Study for one year (1968-69) before accepting a post as professor of mathematics at Rutgers University, where he remained for the rest of his life. He was named chair of the Rutgers mathematics dept from 1975 – 1981, and in 1989 was named first Director of the Science Technology Center in Discrete Mathematics and Theoretical Computer Science at Rutgers, founded by the National Science Foundation. The Center had three supporting groups: Rutgers and Princeton Universities, and AT & T Bell Labs. His major achievement mathematics was his successful classification of finite simple groups. Although many mathematicians (both domestically and around the world) contributed to this area, it was Dr. Gorenstein's efforts that were successful in bringing all simple finite groups under one classification scheme.

Patricia Kenschaft is a professor of mathematics who taught at Montclair State College (MSC, now MSU) for 40 years. She has authored 11 books, to date, and over 50 scholarly articles. She particularly focuses on math anxiety, women, and minorities in mathematics. One of her books, *Math Power: How to help your child love math even if you don't*, is written specifically for the purpose of lessening the math anxiety of many people struggling with math, and their children. She is the founding president of New Jersey Women in Mathematics, Program Chair in 1978-79 of the National Association for Women in Mathematics, and has administered numerous grants funded by the Eisenhower Foundation, the National Science Foundation, and the Exxon Educational Foundation.

Steven Krulik is a mathematics education professor at Temple University, and has authored many books on mathematics education, and particularly in Problem Solving. He was president of the Association of Mathematics Teachers of New Jersey, and editor of the National Council of Teachers of Mathematics 1980 yearbook: *Problem Solving in School Mathematics*. He has trained many K-12 teachers and has supervised dozens of doctoral dissertations at Temple University.

Audry Leef is one of the first and youngest female mathematics educators in the country. After graduating valedictorian from Hoboken High School at 16, she earned her bachelor's degree from Montclair State College in 1943, and began teaching in Millburn High School. Thirsting for more knowledge if mathematics, she went to Stevens Institute of Technology for a masters in mathematics, being the first female to student to attend Stevens. She got her mathematics degree there in 1947. After teaching at Millburn High School and raising a family she was recruited for the Montclair State College's Mathematics Department in 1966, and taught math classes for 26 years, and she is still an emeritus professor in that department. Her specialty is math anxiety and encouraging women in the mathematical sciences. "I always tell women that if you have any leaning toward mathematics, do it now. Don't put it off." At MSC (now MSU) she has served in just about every leadership capacity imaginable (Alumni Association President, Foundation board member, and campus chaplain).



Evan Meletsky Taught mathematics and mathematics education at Montclair State University from the late 1950's to the 1990's. He taught both undergraduate and graduate mathematics and mathematics methods courses, and inspired many math students in that capacity. Many of his students went on to be mathematics teacher educators, themselves. He was lead author of the Harcourt Math series, writing over a dozen classroom textbooks at the elementary and middle school level. He wrote many other books on mathematics methods and appreciation of mathematics. He was active in the national conferences of both the National Council of Teachers of Mathematics and the Mathematical Association of America. He was awarded the distinguished teacher of the year award from the MAA in 2002.

Joseph Rosenstein is a mathematician who works at Rutgers. The State University of New Jersey. He has been passionate about exposing all levels of students (K – 12) and college to discrete mathematics, which includes graphs, coloring maps, systematic listing and counting, circuits, and trees, among other topics (such as the Towers of Hanoi, the 4-Color problem, and the Traveling Salesman problem). He has written several books on discrete mathematics, and from 1987 to 2017 ran the PreCalculus Conference at Rutgers University.

Max Sobel was an author, teacher, and professor of mathematics education. He taught in Newark school system, becoming very interested in the learning difficulties of average and below-average students. He became nationally known for this specialization and lectured and wrote about it in the 1950s and the 1960s. He taught at Montclair State College. He contributed to the School Mathematics Study Group in Chicago, and he also wrote many mathematics textbooks. He became president of the National Council of Teachers of Mathematics in 1958-59.

Bruce R. Vogeli was a professor of Math Education at Columbia University's Teachers College. He entered service there in 1964 and was made a full professor at the age of 35, the youngest professor to achieve that rank. In 1972 he was named the Clifford Brewster Upton Chair of Mathematics Education, an endowed chair at Teachers College. As the senior author at Silver-Burdett, he authored over 100 mathematics books for all grades, 1 – 12. He published dozens of papers and articles. At Teachers College, he mentored over 2,000 students, and sponsored over 200 doctoral dissertations. Many of his students serve in education ministries in many countries. A world mathematics education leader, he led workshops, symposia, and colloquia at Teachers College, around the US, and in over 10 countries in four continents. Not content to ever be idle, he served in the Peace Corps in Nepal in 1998-2000 (at 68) years of age. Ever committed to International Mathematics Education, he co-edited surveys of mathematics education in the Asia-Pacific region, and in the South America.

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Tom Walsh earned a B.S. in Oceanography and Meteorology from the State University of New York Maritime College in 1975, and sailed in the merchant marine. In 1979, he entered the Peace Corps, teaching mathematics and science in Liberia, West Africa. Coming home to America, he studied at Teachers College, Columbia University, earning an M.A., an M. Ed., and finally an Ed. D. (in 1993) in mathematics education, all the while teaching secondary mathematics in northern New Jersey. Entering college teaching in 1994, he has taught mathematics and mathematics methods in several colleges in the New York metropolitan area, finally accepting a position as a mathematics education professor at Kean University in 1999. At Kean, he coordinated several undergraduate and graduate programs in education, administered two grants, and authored five books on mathematics. He has presented mathematics topics at state, regional, and national Conferences. He served as President of the Association of Mathematics Teachers of New Jersey in 2018. Presently, he is an associate professor at Kean, preparing the next generation of New Jersey educators at all levels (elementary, middle, and secondary).

Using Interactive Whiteboard Applications in the K-12 Mathematics Classroom

Michael Finetti, Ed. D.
Saint Peter's University
mfinetti@saintpeters.edu

Nicole Luongo, Ed. D.
Saint Peter's University
nluongo@saintpeters.edu

Abstract: This article discusses how current K-12 teachers of mathematics can use interactive whiteboards (IWBs) and their applications in face-to-face and remote classrooms. The authors present the history behind using traditional whiteboards, as well as how current teachers of mathematics are using modern IWBs and their tools in the classroom. The article describes the importance of using the features of IWBs to enhance mathematics instruction and meet learning objectives. The authors explain how IWBs can be used for brainstorming and organizing ideas, developing mathematical concepts, and improving mathematical knowledge. The article concludes with sharing links to various free and fee-based online IWB applications and tools. These applications are explained in detail with direct examples on how K-12 teachers of mathematics could use them to meet student learning objectives and goals.

Introduction

Today, K-12 teachers of mathematics require a variety of technological applications; teachers are not simply using a physical chalkboard, and in some cases, they are not in the same room as their students. In the past, teachers stood in front of a classroom of live students with chalk and chalkboard, an overhead projector and paper and pencils; now these teachers are using Zoom or Google Classroom and employing applications such as GeoGebra and Desmos. Presently, teachers and students are expected to work on math problems in the "virtual world" using apps and online forums. How can K-12 teachers of mathematics use current and evolving technology tools to effectively meet their learning objectives?

An interactive whiteboard (IWB) is one solution; with its various applications, the IWB can be used by a teacher to present advanced mathematical theories and concepts in innovative and engaging ways. It can also be used by K-12 students to collaborate and communicate with each other and their teachers. This article will explain the history of the IWB and the theory behind using IWBs; it will show how K-12 mathematics teachers can use the IWB in face-to-face or remote classrooms. Links to various free and fee-based online IWB applications and tools are included.

History of Whiteboards

In recent years, "dry erase whiteboards" remain a proven and popular choice in classroom teaching. Large wall-mounted whiteboards are used to display and share information in front of the entire class, while individual whiteboards can be used in place of paper to quickly copy information or notes as students follow along" (Shaw, 2014, para. 1). In fact, traditional whiteboards replaced many outdated chalkboards (LearnCube, 2021). The first set of physical whiteboards used wet and dry-erase markers. These boards were difficult to clean when removing written classwork. Since that time, many traditional whiteboards have been replaced by computer-mediated whiteboards, such as SMARTBoards or Promethean boards, and can be swiped clean by the click of the mouse.

During this evolution, whiteboards were used by teachers during lessons as presentation tools during teacher-led lessons. As the whiteboards became more technical in nature, teachers were able to create a more interactive learning environment for those students who were born and educated in the current digital world. SMARTBoards, Promethean boards and Google Jamboards have been added to many K-12 classrooms (Google, 2021; Mumford, n. d.). Furthermore, the proprietary software coming with these IWBs allow educators to create interactive presentations, quizzes, polls, problems, and projects. These companies provide teachers of mathematics with various lesson materials, including mathematical graphics and images that can be used in both face-to face and remote classes. Many of the current IWB applications allow teachers to share mathematics materials with students, which means that all learners can access and interact with lesson materials using any web-enabled computer or device. As Betcher and Lee (2009) explained, "Unlike the chalkboard, the IWB is not a tool for the teacher; it is a resource to be used by the whole class" (p. 58).

In today's world, various instructional technology tools such as IWBs have been developed to assist in connecting digital learners and educators (Luongo, 2015; Prensky, 2010). "To efficiently do this in an online environment, technology tools, resources, and platforms such as online collaborative groups (for example, breakout rooms in Zoom or Google Meet), digital whiteboards (such as the whiteboard sharing tool in Zoom and Jamboard in G Suite), online collaborative tools (for example Padlet, Pear Deck, Google Docs, Slides, or Drawings) should be included in daily instruction" (Wilder, 2020, p. 27). These applications are critical for use in today's ever-changing K-12 mathematics classroom. As Yildiz explained, "Especially during unprecedented times such as natural disasters, political climate changes, and global pandemics where schools aren't able to operate face to face teaching, online tools can help in making the distance gap less apart by making learning more engaging and creative" (2020, p. 33). This need has led many educators to use IWBs, which were the first electronic instructional technology designed primarily for use by teachers (Betcher and Lee, 2009).

The defining feature of the IWB is interactivity. IWBs are no longer the blackboard of the past that sat in the front of the room filled with teacher-led equations and homework announcements. Betcher and Lee (2009) asserted:

> The first revolutionary teaching tool—the humble blackboard—found its way into classrooms back in 1801 and had a profound impact

> on the nature of teaching over the next 200 years. The blackboard became synonymous with the traditional classroom and, along with shiny red apples, is still seen as a stereotypical symbol of education. The interactive whiteboard—or IWB—has the potential to be the second revolutionary teaching tool. Just as the blackboard was seen as a key part of nineteenth- and twentieth century classrooms, the IWB has the capability to become synonymous with the new digital classrooms of the twenty-first century. Despite its relative newness, the IWB exhibits the same capacity to fundamentally change—and indeed revolutionise—the nature of teaching (p. 1)

If teachers choose to ignore the interactive aspect of IWB technology, they may as well not use one. Teachers who use IWBs can take advantage of the interactivity factor by designing lessons that allow learners to work with the IWBs, both physically and mentally. Many of today's IWBs can be used synchronously and asynchronously in both in the physical K-12 classroom as well as in the remote classroom.

Indeed, IWBs are a form of digital instructional technology that K-12 mathematics teachers can employ in their everyday teaching--whether it be face-to-face instruction or via distance learning. Collins (2019) described the benefits of using IWBs while supporting students' learning of mathematical concepts, which are dependent on the software and digital tools that can be used with IWBs. Maher et al. (as cited in Collins) suggested "greater emphasis should be placed on the digital resources teachers choose to incorporate into their instruction to improve student achievement" (p. 181). As was discussed, planning is key in conjunction with using the IWB and its applications. The planning must focus on obtaining the student learning objectives or goals associated with each lesson or unit. Moreover, Betcher and Lee (2009) stressed that IWBs can accommodate various teaching and learning styles and can be used to support whole-class, small-group and one-on-one teaching.

Interactive Whiteboards (IWBs) in the Mathematics Classroom

Mathematics teachers are cognizant of the various forms of tasks their students need to complete. How technology is integrated into K-12 mathematics instruction can be influenced by the relationship between teachers' mathematical knowledge and skills, pedagogical practices and a recognition of how to use the technology (Koeler et al., as cited in Collins, 2019). Each level of mathematics instruction often requires a different method of teaching. Stein and Smith (as cited in De Vita, Verschaffel, & Elen, 2018) distinguished between mathematical tasks at varying levels.

First, there are low-level tasks such as memorization, practicing facts, rules, formulas and definitions. These are often tasks that do not require higher-level thinking skills. Students are asked to apply procedures without connections to concepts or meaning, such as the use of formulas, algorithms or basic rules. Secondly, there are high-level mathematical tasks where students connect certain procedures with concepts, higher order thinking skills, and meaning. During these high-level tasks, students are often asked to apply formulas, algorithms or procedures to authentic, real world mathematics problems. When completing these high-level tasks, K-12 mathematics students are exploring complex mathematical thinking and reasoning activities such as making and testing conjectures, framing problems, and looking for patterns.

As teachers attempt to teach higher-level mathematics, they will need a way to explain certain concepts and meet the needs of today's digital learners. IWBs can be used to achieve this result, but teachers need training in order to be able to use the tools. A common theme throughout the IWB literature is that teacher proficiency with the technology is a key factor in determining the effectiveness of its application in classrooms (Holmes, 2009). "Teachers' own learning experiences with technology is a major determinant of the extent to which they will incorporate technology when teaching" (Crison, Lerman & Winbourne, as cited in Holmes, p. 355). Therefore, it is recommended that all teachers of mathematics are trained in the use of various IWBs before using the technology with their students.

> Having an IWB gives you the freedom to follow where kids take the lesson. Instead of having to 'come back to that later' because you haven't got resources or items to more fully dive into those ideas, you can utilise the Internet or multimedia or other resources to delve into the kids' ideas then and there and be able to look into them more closely. The kids don't just get to see images; they can manipulate things such as with Google Earth when exploring the world, or move pieces of text to make up stories. With an IWB, students can have a lot more ownership over lessons and the direction they take. (Signal, as cited in Betcher and Lee, 2009, p. 59)

Interactive Whiteboard (IWB) Applications

There are several IWB applications that can help K-12 teachers of mathematics present materials in innovative, 21st century ways. In fact, there are so many new and innovative IWB applications that choosing one can be daunting (Wilder, 2020). "Interactive whiteboard apps take advantage of touch controls, along with easy sharing and collaborating features to help teachers and students think, organize, and create everything from mind maps to lessons to presentations" (Common Sense Media, 2021, para. 1). As you choose an application, there are some factors that you may need to consider: ease of use, price, and usability. Here are some of the most popular IWBs that many teachers of mathematics are using in the K-12 classroom.

Miro: Miro https://miro.com/ is an IWB application that allows students and teachers to create diagrams, draw freehand, capture ideas on sticky notes, add snapshots of webpages, and leave comments for other users (Miro, 2021). The Miro whiteboard can be used both synchronously and asynchronously as teachers work with students. Yildiz (2020) explained that one key benefit of Miro compared to other online interactive whiteboard tools is that it has an option of a real-time integrated video call. This added real-time collaboration with a video call option can mimic an in-person class with teacher and learners. Using the Miro application, a teacher can create online collaborative groups among learners and give ongoing verbal and written feedback on their work by either real-time video or by adding comments through the use of post-it notes. Another Miro feature that mathematics teachers may enjoy is the ability to upload equations, pictures, and other multimedia to enhance student learning.

Previously, there was a whiteboard application called AWWBoard (https://awwapp.com/), which is now associated with Miro. AWWBoard was a touch-friendly online IWB application that allows teachers and students to create drawings and collaborate in real time (Common Sense Media, 2021). AWWBoard (and now Miro) provided K-12 mathematics teachers a way to share equations and solve problems with students. The AWWBoard allowed mathematics students to become creators and teachers to act as mentors (AWW A Web Whiteboard, 2021). Some of the features of AWWBoard include creating, saving, moving or duplicating a drawing with color and highlighting tools. Teachers can change, cut or delete any items posted, or clear the entire page. Users of this whiteboard application have the option to add shapes, write text, and add post-it notes. Another significant feature of this application is the ability to upload images, .pdf files and Powerpoint presentations during a synchronous online mathematics class. The AWWBoard (now Miro) also allows users to choose from templates such as Coordinate Plane, Brainstorming or KWL Chart. Teachers can embed the interactive whiteboard on their class website or create a QR Code for easy access.

Explain Everything Whiteboard:

Explain Everything Whiteboard (https://explaineverything.com/) is an online IWB application that allows educators and students to work collaboratively to create interactive projects and videos (Explain Everything Whiteboard, 2021). One feature of this IWB is that students can create video summaries of their mathematics problems in which they can present the most important things they learned and what questions they may still have. Another facet of Explain Everything is that it allows students and teachers to see and hear themselves while using the application. Furthermore, when using the Explain Everything application, learners can assess and reflect on how they are reaching a learning outcome or completing a high-level task. This IWB also allows the teacher to provide in-depth guidance by sending formative and summative video feedback to each student or to the group. The application allows the teacher to annotate and narrate over any document, media, or presentation on a recordable whiteboard. This is a versatile tool for creating multimedia presentations, too (Common Sense Media, 2021).

Whiteboard.fi: Whiteboard.fi (https://whiteboard.fi) is an IWB that allows all students to have their own digital whiteboard, where they can draw, write text, make notations on images, and add mathematics equations (Whiteboard.fi, 2021). Whiteboard.fi gives teachers the option to see all of their students' whiteboards in real time, so they can follow their progress, while the students only see their own whiteboard and the teacher's whiteboard. It is suggested to use Whiteboard.fi as an instant formative assessment tool for the classroom, providing teachers with live feedback and immediate overview of all students. Another bonus of using this IWB is that it allows all users the ability to insert math symbols, expressions and equations using the math editor, where they can type using a graphical interface or insert LaTeX code. Teachers of mathematics who use physical whiteboards in a class when formatively assessing their students can replicate this experience in a digital environment. When using Whiteboard.fi, accounts are not needed and the IWB will not store teacher or student data. However, teachers may download their work for future reference. Every time a mathematics teacher uses the program, the teacher will set up a new class and create a new whiteboard. Once a new class is created, teachers must provide the direct link to their students before entrance. There is also a waiting room option for security measures. Students can then join the class with their very own personal whiteboard on their screen. As more students enter the room, additional student whiteboards will be seen in the room. The student must toggle the teacher whiteboard in order to see the teacher board.

Several mathematics features exist that make Whiteboard.fi different from others (New EdTech Classroom, 2020). There is a drawing tool, text tool, and mathematics language tool. Custom and premade shapes are available as well. There is an option to bring images directly on the board, along with a zoom in and zoom out tool to realign, resize or drag images. Additionally, the background can be set up as graph paper to write directly on top of it. The mathematics equation editor allows the teacher to include different math equations such as fractions and exponents. The push feature allows the teacher to share their whiteboard with their students. The teacher can view all individual whiteboards at the same time and see their progress in real time. Mathematics teachers can fix a background and have students write directly on top of the background. Teachers can share their whiteboards with individual students or wipe all whiteboards clean at the same time. All Whiteboard.fi whiteboards can be downloaded as images or .pdf files and cleared before starting a new lesson.

Google Jamboard: Google Jamboard (https://gsuite.google.com/jamboard) is another versatile whiteboard application that integrates with all Google Suite applications (Common Sense Media, 2021; Google, 2021; Wilder, 2020). This tool can be used for class meetings and presentation purposes. G Suite users are able to directly access it and share it with other G Suite users. Just like a traditional whiteboard, Jamboard allows students to add images from a Google search, save work to the cloud automatically, and use a handwriting and shape recognition tool, and draw with a stylus but erase with a finger. This feature is particularly useful for a mathematics teacher who needs to explain topics using equations or mathematical notation. Mathematics teachers can use the Jamboard to jot down symbols or symbolic expressions that are intended to have a semantic meaning. The Jamboard can be used by students to show their math work as they have access to pen tools (Abrahamson, 2020). The teacher can post a problem on a “sticky note” and the students can show their work and solve the problem right on the Jamboard page.

Additionally, a Jamboard can have multiple pages for multiple mathematical problems. Moreover, Google Jamboard makes learning visible and accessible to all teachers and students who are added to the "Jam" session. Teachers and students can work cooperatively to solve math equations or share information with each other in real time. Finally, Jamboard allows students and teachers to collaborate from anywhere. Teachers of mathematics can engage all students in the learning process by using a "Jam" session, whether working together in a traditional face-to-face classroom or through distance learning.

Conclusion

In conclusion, IWBs can be useful tools for teachers of mathematics for various reasons. Mainly, these whiteboards and their applications are used by teachers to present advanced mathematical equations and formulas to students in a digital format. Also, teachers can design instruction using IWBs in a way that students can virtually collaborate and communicate with each other. The IWBs can be used by students for brainstorming and organizing ideas while also using the boards to develop specific mathematical concepts, improve mathematical knowledge and explore ways to accomplish various tasks. With proper training and professional development, teachers can employ IWBs and their applications to increase student engagement and help their students learn mathematics in an innovative and impactful way.

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Michael Finetti, Ed.D., is an Associate Professor of Education at Saint Peter's University in Jersey City, NJ, where he serves as the Director of Special Education Programs in the Caulfield School of Education and oversees the Endorsement Program for Teacher of Students with Disabilities Certification. He focuses on instructional strategies, educational approaches, placement alternatives, adapting the learning environment, differentiating instruction, co-teaching arrangements, assistive technology and various performance assessment options for students with disabilities. His own research discusses the effectiveness of collaborative instruction and assistive technology for students with disabilities in inclusive settings.

Nicole Luongo, Ed. D. is a Professor of Education and Director of Distance Learning at Saint Peter's University in Jersey City, NJ. Currently, she teaches a blend of online, hybrid, and face-to-face graduate and undergraduate education courses. Previously, Dr. Luongo worked as an elementary teacher in New Jersey. She taught a variety of subjects, including mathematics, to elementary (K-6) students. She daily utilized computers and other forms of technology.

The Absence of Resources for Teaching Mathematics to Students with Autism: Implications for Teachers

Cassi Zappala
Little Silver Public Schools, NJ
Zappala.cassi@gmail.com

Nicole Hansen, Ph.D.
Farleigh Dickinson University
nhansen@fcu.edu

Abstract: The purpose of the present study was first to review the extant literature to uncover best practices related to teaching mathematics to K-12 students with autism in inclusive classrooms. A second purpose was to systematically review the availability and quality of resources for teaching mathematics to students with autism in three practitioner-oriented journals written for teachers of mathematics. Results of the literature review indicate several recommendations for teaching mathematics to students with autism: schema approaches, using manipulatives, using graphic organizers, using the CRA framework, and teaching metacognitive monitoring. Further, results of the systematic review reveal that just one of the included articles mentions autism and that only a small percentage of included articles mention any type of disability. While a higher percentage of articles included research-based strategies that are shown to be effective for teaching mathematics to students with autism, many of these articles still do not provide the level of detail a teacher would need to implement the strategies. Implications for teachers, students, and the field of education more broadly are discussed.

Introduction

Autism is currently recognized as the fastest growing developmental disability in the United States (Gargiulo, 2014) and recent studies estimate the prevalence of autism in eight-year-old children to have reached 14.5 per 1,000 (Christensen et al., 2018). One concern facing educators today is how to effectively educate students with autism, particularly in mathematics. While autism alone is not necessarily a predictor of low performance in mathematics (Chiang & Lin, 2007), comorbidity of autism and learning disabilities is sometimes observed in school-aged children (Mayes & Calhoun, 2003) and research suggests that the percentage of individuals with autism who also have a learning disability may be as high as 75% (O'Brien & Pearson, 2003). Given that a successful high school mathematics education is predictive of success in college and beyond (National Mathematics Advisory Panel, 2008), it is imperative that students with autism develop strong mathematics skills while in school.

Literature Review

Autism: Definitions and Criteria

Autism is a developmental disorder that affects approximately one to 1.5 million individuals in the United States (Gargiulo, 2014). According to the guidelines set by the *DSM-5* (2013), autism is characterized by several diagnostic criteria: deficits in communication and social interaction, restricted and repetitive patterns of behavior, presence of symptoms in the early developmental period, and significant impairment in social, occupational, and other areas of functioning (American Psychiatric Association, 2013).

Autism and the Mathematics Classroom

Although a diagnosis of autism in itself does not suggest that a student will perform poorly in school (Chiang & Lin, 2007), many students with autism struggle with mathematics. One study evaluating the academic achievement of students with autism found that although some students performed at average or above-average in mathematics, other performed far below average (Keen, Webster, & Ridley, 2016). Research suggests that appropriate instruction in mathematics early on can play a pivotal role in life outcomes for students with autism (Gargiulo, 2014).

Intervention Strategies for Students with Autism in Mathematics

A review of the existing literature uncovered five main strategies that are appropriate for students in kindergarten through twelfth grade with autism who are educated either part-time or full-time in inclusive classroom settings. Consequently, most of the research presented will be specifically concerned with teaching strategies for students with mild or moderate autism.

Schema approaches. Some studies indicate that schema approaches may be useful tools to help students with autism learn mathematics. Kasap and Ergenekon (2017) defined a schema approach as having four steps: create a need, set an example, provide students with opportunities for guided applications, and allow students to independently apply what they have learned (Kasap & Ergenekon, 2017). Studies illustrate that using schema approaches can improve the performance of students with autism in areas such as addition and subtraction (Rockwell et al., 2011) and verbal problem-solving skills for comparison-type problems (Kasap & Ergenekon, 2017).

Manipulatives. In an overview of teaching mathematics to students with disabilities, Gurganus (2017) states that both concrete and virtual manipulatives can be used to improve performance across the mathematics curriculum. Virtual manipulatives are an effective tool for teaching students with disabilities in a variety of different areas including fourth-grade fractions, single and multi-digit subtraction, and high school geometry. Additionally, the opportunity to use manipulatives is a motivating factor for many students (Gurganus, 2017).

Graphic organizers. The effectiveness of graphic organizers for facilitating the learning of mathematics for students with autism is questionable. A study by Delisio, Bukaty, & Taylor (2018) tested the effectiveness of a specific type of graphic organizer to help students with autism solve a variety of math problems, but the results did not suggest that the graphic organizer was helpful for all students with autism (Delisio, Bukaty, & Taylor, 2018). However, the benefits of graphic organizers have been widely discussed for students with disabilities more generally (Gurganus, 2017).

Metacognitive monitoring. Researchers Maras, Gamble, and Brosnan (2017) designed an experiment to investigate the impact of a computer-based program that encourages students to practice metacognitive monitoring, which is the self-regulation of learning made possible by being aware of one's own thought process. The researchers found that metacognitive support benefitted both students with autism and their typically developing peers, but had a much more profound impact on the mathematics performance of students with autism (Maras, Gamble, & Brosnan, 2017).

CRA framework. Another strategy that may be effective for students with autism is the concrete-representational-abstract (CRA) sequence. This framework takes students through different modes of interacting with objects (concrete), looking at visuals (representational), and understanding symbols (abstract) (Gurganus, 2017). Students in a 2016 study who took part in a video-based CRA framework intervention experienced an increase in skill acquisition and maintenance (Yakubova, Hughes & Shinaberry, 2016).

Limitations

While the studies discussed above provide insight related to effective teaching strategies for students with autism, limitations in this body of literature exist, including the low number of existing studies focused of how students with autism learn mathematics and the small sample size of many of the studies.

The Present Study

The purpose of this study was to investigate the availability and quality of resources published in practitioner-oriented mathematics journals that can be used by general education mathematics teachers to teach students with autism. We hypothesized that while many of the articles published in practitioner-oriented mathematics journals may be relevant for teaching mathematics to students with autism, these articles may not provide clear enough directions for general education teachers to actually apply the strategies to students with autism. As such, the study asked the following research questions:

1. What percent of resources for general education mathematics teachers published in practitioner-oriented mathematics journals are relevant for teachers of students with autism?

   a. Do the relevant articles mention disability explicitly? If so, do they mention autism spectrum disorders specifically?
   b. Do the relevant articles mention evidence-based teaching techniques for students with autism? Specifically, do the relevant articles mention schema approaches, manipulatives, graphic organizers, metacognitive monitoring, or the CRA framework?

2. What is the quality of the resources for general education mathematics teachers published in practitioner-oriented mathematics journals are relevant for teachers of students with autism?
   a. What grade levels do these resources address?
   b. What is the quality of these resources?

Methodology

This study took the approach of a systematic review, which is a thorough review of all relevant publications meeting a specific set of criteria (The Himmelfarb Health Sciences Library, n.d.). To ensure that the review was systematic, we considered the following: search procedures, criteria for inclusion and exclusion, and a data extraction and coding system.

Search Procedures

First, a list of journals that target general educators who teach mathematics for grades K-12 was generated. Ultimately, due to full-text library availability, all articles published in 2016 from both *Mathematics Teacher* and *Teaching Children Mathematics* were originally included in this review. Given that the two journals were eventually consolidated into a new journal, *Mathematics Teacher: Learning and Teaching PK-12*, we later reviewed all articles from this journal published in 2020 and added them to the results. Criteria for inclusion and exclusion (listed below) will be followed. Specifically, we only examined articles that targeted general educators of students in K-12 (not researchers or teacher educators).

Criteria for Inclusion and Exclusion

Several criteria were used to determine whether or not an article was relevant to the present study. All included articles were published in 2016 or 2020, focused on how to teach mathematics, and appeared to have an intended audience of practicing general educators teaching mathematics to students in grades K-12. Articles not meeting all of these criteria were excluded. Figure 1 details the coding system fully. All articles that were deemed appropriate to include and relevant to the study using the criteria in Figure 1 were analyzed qualitatively and are discussed further in this paper.

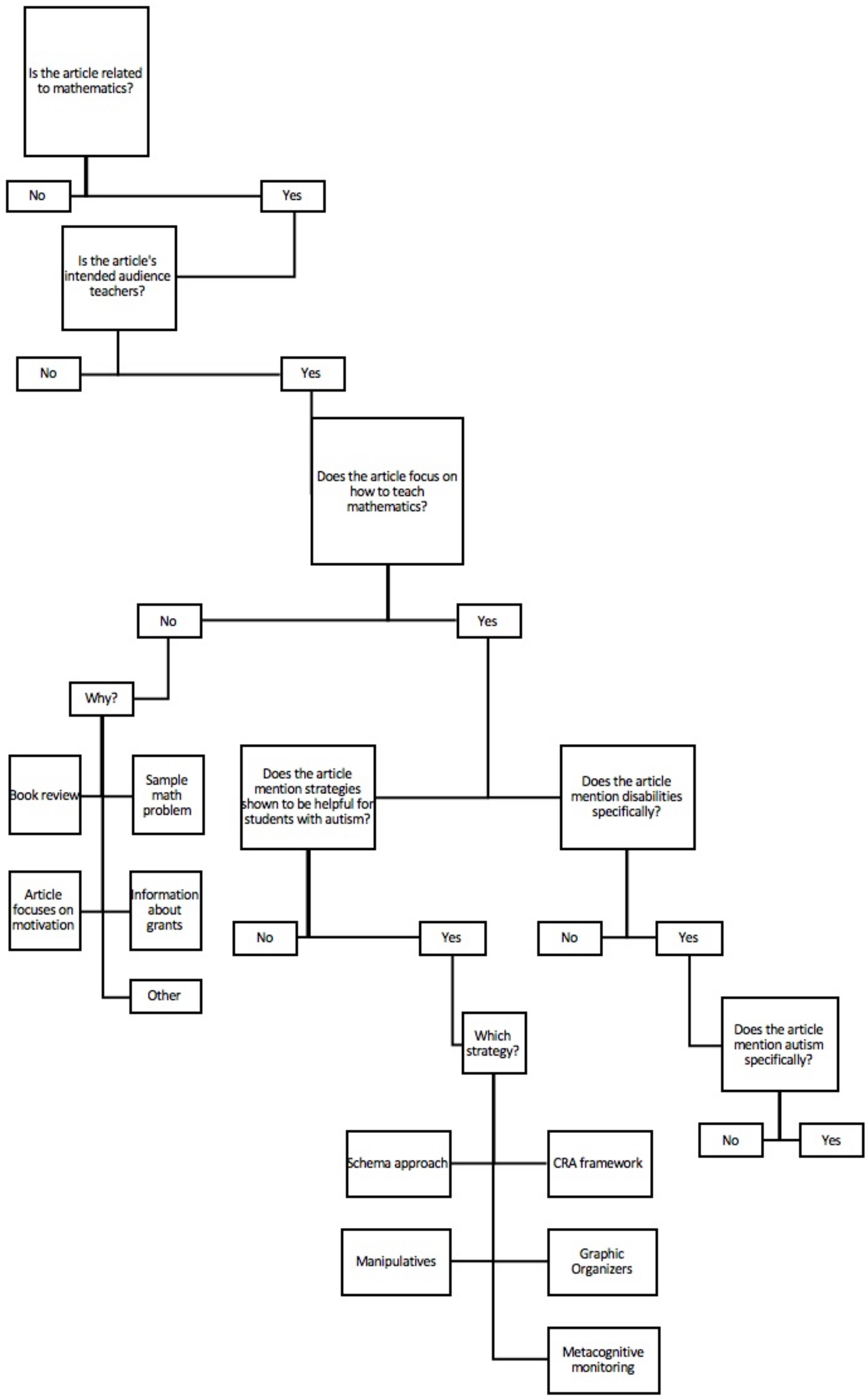

Figure 1. Article coding system.

Results

What Percent of Resources for General Education Mathematics Teachers Published in Practitioner-oriented Mathematics Journals are Relevant for Teachers of Students with Autism?

An evaluation of all 505 articles published in *Mathematics Teacher* and *Teaching Children Mathematics* in 2016 and *Mathematics Teacher: Learning and Teaching PK-12 in 2020* found 303 articles that met the criteria for inclusion in the study. Of the 303 included articles, 75 articles, or 24.8%, either mentioned disabilities or included at least one of the selected evidence-based strategies for teaching students with autism (schema approaches, manipulatives, graphic organizers, metacognitive monitoring, or the CRA framework).

Table 1

Criteria	Number of Articles	Percent of Included Articles
Mentions either disabilities or selected evidence-based strategies (or both)	75	24.8%
Mentions evidence-based strategies	69	22.7%
Mentions any type of disability	13	4.3%
Mentions autism	1	0.0%

Do the relevant articles mention disability explicitly? If so, do they mention autism spectrum disorders specifically?

Of the 303 articles included in the study, one article mentioned autism. However, there were 13 articles (4.3% of articles) that mentioned either disabilities in general or a specific disability.

Do the relevant articles mention evidence-based teaching techniques for students with autism? Specifically, do the relevant articles mention schema approaches, manipulatives, graphic organizers, metacognitive monitoring, or the CRA framework?

Of the 303 articles included in the study, 69 articles, or 22.7% of articles, included the selected evidence-based teaching methods (see Table 1).

Content of Articles

What is the Quality of the Resources for General Education Mathematics Teachers Published in Practitioner-oriented Mathematics Journals are Relevant for Teachers of Students with Autism?

What grade levels do these resources address? Of the 13 journal articles that mentioned disabilities, three were included in a journal that targeted secondary educators, four were included in a journal that targeted elementary educators, and six were included in a journal that targeted K-12 mathematics educators. Of the 69 articles that included the selected strategies (schema approaches, manipulatives, graphic organizers, metacognitive monitoring, or the CRA framework), seven were intended for secondary educators, 26 were written for elementary teachers, and 36 were intended for K-12 educators more generally.

What is the quality of these resources? The resources that were determined to be relevant for teaching mathematics to students with autism vary greatly in quality. For example, the article that focused on teaching a student with autism (Yeh, Sugita, & Tan, 2020) discussed promoting critical thinking and using multiple representations with the student, but did not specifically mention one of the selected strategies. On the other hand, a different article (Hord et al., 2016) that mentioned students with disabilities provided teachers with an in-depth explanation of how a selected strategy (a schema approach) can be used to teach mathematics to students with disabilities.

Some of the articles that included students with disabilities simply mentioned disabilities in passing and did not provide any mention of how to actually teach students with disabilities. For instance, one article cited a student who was born without an arm as the inspiration behind a class project, but did not mention teaching strategies for teaching students with disabilities (Bush et al., 2016). In another, the term "disability status" was included in a list of ways that a classroom can be diverse, but the remainder of the article did not mention disabilities again (Rubel, 2016).

Among articles that mentioned the selected strategies, few included clear instructions (i.e., steps or directions that can be replicated) for using strategies that happen to be effective for educating students with autism. Very few articles explicitly made the connection between the teaching strategy and how it can be used to teach students with disabilities. The majority of articles meeting these criteria simply mentioned the selected strategies as possible tools, but do not explicitly explain how or when these tools are useful. In general, the majority of articles did not provide enough information for teachers to effectively use the strategies to teach mathematics to students with autism.

Discussion

The present study investigated the availability and quality of resources published in practitioner-oriented mathematics journals that can be used by general education mathematics teachers to teach students with autism. After conducting a systematic review of all journals published in *Teaching Children Mathematics* and *Mathematics Teacher* in 2016 and *Mathematics Teacher: Learning and Teaching PK-12* in 2020, it was concluded that only a small percentage (just 25%) of articles about teaching mathematics are relevant for teaching students with autism. Even among the articles that were determined to be "relevant," a majority of articles still do not present the information in a way that provides teachers with a clear idea of how to implement the practices in their classrooms.

It is concerning to think that there is a shortage of resources in any area of teaching. However, it is particularly worrisome that the shortage identified in the present study is related to resources for teaching mathematics to students with autism. Autism is currently recognized as the fastest growing developmental disability in the United States (Gargiulo, 2014), yet it is mentioned only one time in the 303 articles included in the present study. It is especially important to consider the need for resources specifically related to teaching mathematics to students with autism given that many students with autism struggle with math (Keen, Webster, & Ridley, 2016).

Approximately 39.9% of students with autism are educated in general education classrooms (Gargiulo, 2014). However, the results of the present study show that the needs of students with autism are not being adequately addressed in practitioner-oriented journals about teaching mathematics. This has implications for both students with autism as well as other students. A shortage of resources in any area creates a need for teachers to spend extra time searching for and creating resources, likely taking away from time that could be spent maximizing these resources. If this is the case, there is the implication that students may receive less quality resources and instruction from their teachers.

There are limitations that should be considered when reviewing the results of the present study. First, the study only looked at three practitioner-oriented journals. All three of these journals are published by the same professional organization (The National Council of Teachers of Mathematics) and may not be representative of all practitioner-oriented journals about teaching mathematics. Second, the methodology of the present study exclusively reviews the selected major practitioner-oriented journals and does not include searching databases, which may yield different results. Finally, it should be considered that the research-based strategies discussed in this study are not the only strategies that are effective for teaching mathematics to students with autism. There are many other teaching strategies that may be effective, yet need additional research at this time.

Conclusions and Directions for Future Research

The lack of resources for teaching mathematics to students with autism discovered in this present study highlights a need for changes. First, there must be more research conducted on how to effectively teach mathematics to students with autism. The research on this topic at the time of this writing is very limited.

Second, practitioner-oriented journals must include more resources for teaching mathematics to students with autism and other disabilities. One way to see this goal accomplished may be to encourage teachers who have experience working with students with disabilities and are familiar with current research to contribute more resources to practitioner-oriented journals. Given the time constraints that teachers likely face, it could be useful for school districts to provide incentives for teachers who are actively involved with research. Contributions to research from teachers who have experience in the classroom could be invaluable to the future of students with disabilities.

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Dr. Nicole Hansen is an Associate Professor of Special Education at Fairleigh Dickinson University, where she teaches graduate and undergraduate courses in special education and is the Director of the Learning Disabilities program. Her research focuses on helping students with learning difficulties succeed, both by deepening our understanding of mathematics learning disabilities and by investigating how teachers conceive of and engage with disability.

Cassi Zappala earned her Bachelor of Arts in Mathematics and Master of Arts in Teaching from Fairleigh Dickinson University. She currently teaches Elementary Special Education in Little Silver Public Schools.

Five Fun and Educational Activities for an Introductory Statistics Class

Rebecca Conley, Ph.D.
Saint Peter's University
rconley@saintpeters.edu

Abstract: Incorporating learning activities into an introductory statistics class can improve student learning. There is a wealth of material suggesting many different activities, and a professor could easily find more activities than would fit in a single course. As a new professor I found the options overwhelming, so in this paper, I relay my five favorite activities to do over the semester. The objective of this paper is not to introduce novel activities, but instead to offer a short, accessible list of highlights and to provide a jumping off point to other statistic professors who want to incorporate active learning in their classes.

Introduction

I have been teaching statistics for three years, and as a new statistics professor, I heard early and often about the importance of incorporating activities into my classes. Students learn better in STEM courses that devote some of the class time to active learning; and active learning greatly benefits minority students, students from disadvantaged backgrounds, and female students (Freeman, 2014). There are many resources available, such as (Gnanadesikan, 1997) and (Gelman, 2017). In fact, the number of suggested activities can be overwhelming. In this paper, I share my five favorite activities that I use in introductory level statistics classes.

Sampling Techniques

People often use the word "random" to mean haphazard, strange, or erratic. As they come into class with this definition in mind, students often feel confused when we start talking about random samples. In my class, we do an activity focused on sampling using the random rectangles sheet, which is readily available with a quick internet search. I first heard about the random rectangles in Carolyn Cuff's minicourse *Teaching Introductory Statistics GAISE 2016* (Cuff, 2017). Dr. Cuff cited (Scheaffer, 2013) for this activity, which typically takes about 25 minutes.

The random rectangle sheet has 100 numbered rectangles of different sizes, and the objective is to find the mean area of the rectangles. First, I ask my students to select five typical rectangles, without elaborating on what *typical* means. Then they calculate the average size of the rectangles in their sample. All the student share their results with the class, and I record the numbers on the board.

Next the students select a simple random sample of five rectangles. They use the random number generator Randomizer.org, which works well on smartphones. Once they have a random sample of numbers from 1 to 100, they use these as the ID numbers to select the rectangles. Again, they calculate the average size of their sample, and share the results. During this step, it's a good idea to remind the class they need to average the areas of the rectangles, not the ID numbers. I normally ask them to work in pairs, at least while selecting the sample, as this minimizes the difficulty that some students have going from ID numbers to areas.

Invariably, almost all of the students' estimates from the nonrandom samples are larger than the parameter $\mu = 7.42$. The simple random samples result in some estimates that are too large and some that are too small, but they are typically centered around the parameter. We discuss that when using a convenience sample, some groups are overrepresented and others are under-represented. In this case, the larger rectangles are overrepresented because they are easier to see and thus selected more often. This affords us the opportunity to discuss sampling bias. There are many variations on this activity, and it would be easy to incorporate other sample techniques such as systematic or cluster sampling.

Experimental versus Classical Probability

This is a very simple activity that takes less than five minutes. It can be used when introducing the definitions of relative frequency (or experimental) probability and classical (or theoretical) probability. I ask the students for the classical probability of getting tails when flipping a coin. They quickly arrive at ½. We briefly go through where this number came from. Then, I ask them for the relative frequency probability of getting tails when flipping a coin. Some students will again give ½. I ask them what is wrong with this answer. I wait for someone to realize that we cannot think our way to a relative frequency probability, we need to do a procedure and make some observations!

I pass out some coins, and with the help of some volunteers, the class calculates the relative frequency probability. We compile all the results and calculate one answer. Almost always, it is a least a little different than 0.5. This gives us the opportunity to discuss that the relative frequency probability and the classical probability result in different answers, and this serves as an introduction to the law of large numbers.

Confidence Intervals

To help the students form a more intuitive understanding of confidence intervals, I do this activity before introducing any formulas. The main idea is for students to answer trivia questions using intervals instead of single number guesses (Wang, 2019). We relate these interval answers to confidence intervals. This activity takes about 10 to 15 minutes (and it can be extended with more trivia questions).

First, we produce a 100% confidence interval for the number of full-time professors at our university. I ask the students to help me come up with that interval, telling them that we'll make the interval larger until everyone is 100% sure that the correct answer is within the interval. I guide them through process, for example, saying "there must be at least one professor at this school, I'm standing here". We always end up with an interval that is ridiculously large: something like, there are between 1 and 10,000 full-time professors. Note, there are only about 2,500 undergraduates at our university. I ask the students if this estimate is useful, and they come to the conclusion that clearly it is not.

Next, the students develop a 50% confidence interval for the number of professors at our university, in that, 50% of the class thinks the interval contains the correct answer. It is usually much smaller and more useful. I use this opportunity to discuss confidence versus precision.

Finally, the students create a series of confidence intervals based on trivia questions I pose to the class. We go one question at a time and share the results as we go. I use 50% as a confidence level, but other levels could be used as well. The aim is for half the class to have the correct answer in the interval and half to not. The students usually need to be reminded several times that ‘getting it wrong’ half the time is actually the correct outcome. I take some of the trivia questions from (Wang, 2019) and also add some more local questions, such as “what is the population of our state (in millions)?”, “how many full-time undergraduate students are at the university?”, and “what percent of students are commuters?”. My students tend to struggle with geography questions to the point that they detract from the activity, thus I normally avoid the geography questions from Wang’s list.

One advantage to doing the questions one at a time is that the students can get immediate feedback on the widths of their intervals, and have multiple opportunities to adjust as necessary. Additionally, it gives more flexibility in terms of timing.

Hypothesis Test

The concept behind this activity is to demonstrate that people use empirical evidence to evaluate the validity of their prior assumptions, i.e., we intuitively perform hypothesis testing. This activity takes about 20 minutes. It is adapted from Dr. Cuff’s minicourse, where she cited Eckert’s work (Eckert, 1994).

Before the class, take two identical decks of cards. Open them carefully and preserve the cellophane wrapping; I have found that using an unfolded paperclip to open the bottom works well and cheaper cards (not Bicycle brand) are easier to open. Put all the red cards in one deck and all the black cards in the other. Reseal the packages so that they look like new. I will describe the activity assuming that I have brought the pack with all red cards to class.

I do this activity after the class has already learned the basics of hypothesis testing. However, when I begin it, I do not mention hypothesis testing. I say that we need to review some probability concepts and that we will play a game to do so. Let the students know that the objective of the game is to be the first student to get a black card, and the winner will get a piece of candy. You can ask what the probability of getting a black card is, and write the correct answer on the board, $p = 0.5$.

Open the cards, mention it’s a new deck, take out the jokers, and shuffle the cards several times. Ask for a volunteer and let him or her come to the front and draw a card. It will be red. Shuffle, and ask for another volunteer, again the card will be red. Repeat. Normally, there will be some groans and laughs, and more students will want to try their luck. Typically, by card number four or five someone will say “it’s a trick deck” or “I bet they’re all red”. Remind the students that you opened a new deck and they saw you remove the cellophane. Never admit it is a set up.

Once everyone has had a turn picking a card, feel free to tell them no one wins and all is the candy all yours. Then distribute candy to prevent a riot. Once the students are peacefully and quietly eating candy, you can discuss the important concepts that have just been demonstrated. At this point, all the students will be convinced that the deck is not half red and half black. Ask "what is the opposite of $p = 0.5$?" and then write the correct answer on the board, $p \neq 0.5$. You can ask "do we still think $p = 0.5$?", and the response will be "no". Summarize that in the face of overwhelming evidence, we throw out our original hypothesis and conclude the alternative hypothesis. It is a good idea to stress that we have not *proved* that the whole deck is red, and that we could be wrong. We could have committed a Type I error.

You can also calculate some probabilities, analogous to the *p*-value. For example, if the deck is half black, then the probability of getting four red cards in a row with replacement is 0.0625 and the probability of getting five red cards is 0.03125. In my experience, card four or five is when the first person suggests that the deck is stacked. Since both of these numbers are close to the standard significance level of 0.05, it helps motivates why this value is used as a cut-off point. Whether or not, it should be is a whole other discussion (Wasserstein, 2016).

Computer Lab

Incorporating technology into a statistics class is a common goal. My classes go to the computer lab about eight times during the semester. In the computer lab, normally I start with a short 5- to 10-minute review about the topic and the necessary computer commands. For the rest of the class time (either 40 or 60 minutes), the students then work on a classwork assignment about the topic that we are covering: summary statistics, graphs, probability problems, binomial distribution, normal distribution, confidence intervals, or hypothesis testing.

I set up the classwork assignments on Blackboard, with the option for unlimited submissions selected. For all the topics besides for graphs, I use problems that can be auto-graded, such as calculated numerical answers, multiple choice, or true/false. I walk around the computer lab helping students and answering their questions. If a large number of students are having trouble with the same problem, I will address the whole class as a group. Otherwise, I encourage the students to work in small groups of two or three people; they can answer each other's questions and learn from one another.

It takes time to develop the classwork assignments, and figuring out a reasonable length and difficulty-level of an assignment is a learning process. An easy place to start is to use Google Sheets or Excel for making graphs. Students enjoy personalizing their graphs with different colors and fonts. Furthermore, constructing plots by hand is discourage by the GAISE guidelines (Carver, 2016, page 24). Developing an appropriate title for the graph can be challenging for the students, since it requires them to understand what is being communicated by the graph.

Conclusion

Incorporating activities into an introductory level statistics class can be a great way to improve student learning. In my experience, the students enjoy the activities.

I often refer back to them while lecturing, and have found that students remember them and use them to make connections. While the amount of available material can be overwhelming, a professor does not need to change his/her entire course overnight. Even using a few activities can be rewarding for both the professor and the students. In this paper, I present my five favorite activities, giving an overview, directions, and the approximate amount of time needed.

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Rebecca Conley, Ph.D., is an Assistant Professor in the Department of Mathematics and Statistics at Saint Peter's University in Jersey City, NJ. She earned her Master's Degree in Pure Mathematics from Hunter College and her Ph.D. in Applied Mathematics from Stony Brook University. She primarily teaches statistics courses, including introduction to statistics, regression analysis, and statistical computing in R. Her research interests include numerical analysis and statistics.

Implementing the Center for Applied Linguistics SIOP Interactive Activity Design Template in the Math Classroom

Lisa Rose Johnson, Ed.D.
Rowan University
johnsonlisarose@gmail.cm

Lynn Cruse
Barnegat Township Public School District
Crusrs87@gmail.com

Abstract: Developed in 1999, The Sheltered Instruction Observation Protocol (SIOP) Model is an effective instruction plan to meet the needs of English Language Learners now often referred to as Emergent Bilingual Students. SIOP is often used in classrooms that don't have an English Language support teacher. As the number of emergent bilingual students continues to grow, math teachers can benefit from research-based tools to help in reaching these students. This paper will provide two sample SIOP lesson plans using the Center for Applied Linguistics SIOP Interactive Activity Design Template. This template is an effective way to plan lessons that can meet the needs of all students including emergent bilinguals

Keywords: Center for Applied Linguistics SIOP Interactive Activity Design Template, Emergent Bilinguals, Sheltered Instruction Observation Protocol, Math Lesson Planning

Introduction

The purpose of this article is to introduce the Sheltered Instruction Observation Protocol (SIOP) Model. This model will allow for math teachers to meet the needs of students who receive English Language support, the most recent term for this population of students is emergent bilinguals. Currently New Jersey has seen continued growth in this area. During the 2013-2014 school year New Jersey had 64,208 students who received ESL services. The most data for 2017-2018 showed that 80,693 students received services. In order to support emergent bilinguals, math teachers will need tools to assist in educating this population. Many teacher-preparation programs and school districts in New Jersey as well as the Department of Education promote the Sheltered Instruction Observation Protocol (SIOP) Model. SIOP is an effective tool to educate emergent bilinguals when a TESOL (Teaching English to Speakers of Other Languages) teacher is not able to co-teach or is not responsible for math instruction. This article will provide information on how to plan, execute and assess a SIOP lesson.

What is Sheltered Instruction?

To understand SIOP it helps if you understand the term Sheltered English. Sheltered English was developed by Stephen Krashen (Krashen n.d.) as an instructional approach used to develop grade-level content-area knowledge, academic skills, and increased English proficiency in students who are acquiring English Language Proficiency. Classroom teachers help to shelter and assist students by using clear, direct, simple English. When planning Sheltered English lessons, a variety of effective scaffolding strategies are included to communicate meaningfully to students. The learning activities are developed in ways that connect new content to the prior knowledge of emergent bilingual learners. Sheltered instruction provided emergent bilingual students the opportunity to learn grade-level content instruction from their English-speaking peers, while teachers modify their lesson delivery based on the levels of English Proficiency.

What is the Sheltered Instruction Observation Protocol (SIOP) Model?

The Sheltered Instruction Observation Protocol (SIOP) Model was developed in 1999. The program was developed at the conclusion of intensive observation of Sheltered English teaching across the United States (Echevarria, Vogt, & Short, 2004). The SIOP Model identifies 30 important elements of sheltered instruction under eight broad categories: The SIOP Model is made up of eight interrelated components:

1. Lesson Preparation
2. Building Background
3. Comprehensible Input
4. Strategies
5. Interaction
6. Practice/Application
7. Lesson Delivery
8. Review & Assessment

Delivery of SIOP Lesson

Teachers who use SIOP often have to modify their speech. They may need to rely on visuals, and, when necessary and feasible, content text so that English language learners can grasp important content concepts, facts, and questions. Teachers explicitly teach learning strategies – from teacher-centered to peer-supported to student-centered – so that students develop a toolkit for accomplishing difficult learning tasks. Teachers also provide ample opportunities for students to interact in the target language around purposeful tasks that are meaningful to them.

Step One: Planning A SIOP Lesson Using the Center for Applied Linguistics Interactive Design Template

A quick Google search will reveal several SIOP lesson planning tools, many school districts may even have a SIOP template built into your lesson planning collection software or have existing templates. When I personally plan SIOP lessons I have had success using the Interactive Design Template that can be found on the Center for Applied Linguistics (CAL) webpage. To find this resource please use this link. (https://www.cal.org/siop/lesson-plans/). We follow this format for students in elementary or middle school, who are learning decimals, in a SIOP Lesson from CAL (2009) Practical Applications of Fractions, Percents, and Decimals SIOP Lesson Plan. This lesson plan was initially written by Ivonne Govea as a part of the Two-Way SIOP (TWI SIOP) project conducted at CAL. A TWI refers to lessons that are presented in two languages in a bilingual classroom. This lesson was later adapted for SIOP by Sandra Gutierrez of CAL. I have also made some changes to this lesson plan to fit the Interactive Design Template.

Objectives: This lesson plan utilizes both math content and language objectives.

Content Objectives: The student will (TSW) represent common fractions (1/2, 1/3, 1/4, 1/5, 1/6, 1/8, 1/10) on 10x10 grids.
TSW make connections about the relationship between fractions, percents, and decimals.
TSW apply their knowledge of fractions, percents, and decimals to the real-life task of designing a playground.

Language Objectives: TSW discuss their playground design in small groups.
TSW use key vocabulary to describe relative size and express the same amount in fractions, decimals, and percents.
TSW explain their playground designs to the class, both orally and in writing.

Interactive Activity:

The teacher will share photos of the student's drawings of several school playgrounds. Then the teacher will provide the students with a 10X10 Grid that contains the design of a playground (this could be a design that was created by the teacher or a design that was created by students in previous years). Using highlighters, the playground can be color coded to show divisions and fractions of the area. There can a be red area for swings, a slide or even a rock wall. Another area could have a Four Square court Gaga pit. Since this is a 10X10 Grid frame or 100 squares this makes it easier for discussion and conversations from fractions to decimals. For example, the teacher could say, "The Four-Square court is one fifth or 20 percent or 0.20 of the whole playground area."

Group Configurations/Compositions:

Assign the students roles. One could use the ruler to measure or count. Another student could be the drafter who is drawing the objects on the grid. A third role could be presenter speaker and if you are using groups of four, one could be a timekeeper. As the teams create their playgrounds the teacher can monitor whether students are collaborating and following directions.

Ideas for Academic Interactions and Assessment:

The teacher can monitor and encourage students' use of key content vocabulary and provide feedback as needed. The teacher might also model and use a sentence starter that is available for students to discuss their designs and dimensions. Once groups are finished designing their playground, model how to write their report using the table "My Ideal Playground Design" by filling out one row based on the playground design 10X10 Grid used to introduce the activity.

Academic Language Activity:

The teacher can also review how to represent common fractions in a 10X10 Grid (One half, one third, one fourth, one fifth, one sixth, one eighth, one tenth). Using the posters of equivalent fractions and percents, point out that equivalent decimals and percents can also be represented in the 10 x10 grid. Remind students what resources they can use to help them calculate the equivalent decimal or percent for a fraction (posters, calculators, etc.).

Students can be assessed on how they make the conversions from fractions to percents and decimals based on their design on 10X10 Grid and on a calculations paper. Then, using the table provided in the lesson plan, students prepare their written reports detailing the areas in their playground; the size of each area in fractions, decimals, and percents; the purpose and audience for each area; and their justifications for the size of each area. The lesson plan also includes a rubric that could be used for assessment purposes.

Second Sample Lesson: Congruent Triangles

This second lesson below is adapted from CK-12 Using Congruent Triangles and could be used with middle, secondary and even in higher education settings, where students may have not mastered the concept of congruency.

Objectives:

Content Objectives: TSW state if two triangles are congruent by applying postulates and theorems.
TSW find the distances using congruent triangles
TSW construction techniques to create congruent triangles.
Language Objectives: TSW discuss the 5 theorems.
TSW use key vocabulary to describe sides and angles.
TSW explain the theorems to the class, both orally and in writing.

Interactive Activity: Students will be given manipulatives (toothpicks and gumdrops or mini marshmallows) to create triangles to prove the 5 theorems. They can build the triangles to demonstrate each of these and then label each example.

Name	Corresponding Congruent Parts	Does it prove congruence?
1. SSS	Three sides	Yes
2. SAS	Two sides and the angle between them	Yes
3. ASA	Two angles and the side between them	Yes
4. AAS	Two angles and a side not between them	Yes
5. HL	A hypotenuse and a leg in a right triangle	Yes
6. AAA	Three angles	No—it will create a similar triangle, but not of the same size
7.SSA	Two sides and an angle not between them	No—this can create more than one distinct triangle

Group Configurations/Compositions: There are several appropriate grouping strategies when planning SIOP lessons. Peer learning using a native speaker and an emergent bilingual would be an effective way for the native speaker to model both the academic skill and the academic language.

Academic Language Activity: Geometry allows for academic language to be presented and mastered within math lessons. Math classrooms can benefit from word walls where words can be on display. Teachers can opt to use reinforcers when words from the word wall are used. Teachers can also opt for an interactive notebook that would allow for students to build a math dictionary. Interactive notebooks can be constructed on paper or digitally. Students can also create crossword puzzles which could be used as a study tool, quiz or even a formal assessment. Discovery Education's Puzzle Maker is a helpful tool to create crossword puzzles.

Ideas for Academic Interactions and Assessment: There are many tools that allow for Academic Interactions. Besides using the manipulatives students can provide sentence frames to help students put the concepts into words. These sentences could be used as a formal or informal assessment. They also could be added to the interactive math notebook. The sample lesson from CAL uses simple starters that could be adapted. For example, two triangles are congruent when ____, _____, ____ are _______. These sentences could also be left blank and could include drawings or three-dimensional models. These academic interactions also could be presented to the class either orally spoken, recorded with a tool like Flip Grid or turned into a presentation using software like PowerPoint, Google Slides or Prezi.

Conclusions: Although many SIOP resources can be found on the internet or from many professional development providers CAL has a long history of developing and sharing free resources that can be used to enhance teaching and learning. The SIOP Interactive Activity Design Template is an easy-to-use tool that can be used by all teachers and can be shared among faculty members. In addition to the design template CAL continues to maintain a comprehensive database of free resources and materials for in classroom and professional development when considering or choosing to use the SIOP model to meet the needs of emergent bilingual students.

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Dr. Lisa Rose Johnson, Ed.D., currently teaches ESL in Egg Harbor Township Public Schools and works at Rowan University. She currently holds two executive positions within the American Education Research Association (AERA), Vice-Chair for the Special Interest Group #96 Inclusion and Accessibility in Educational Assessment Secretary/Treasurer of SIG #126 Teacher as Researcher (TAR).

Mrs. Lynn Cruse is an English as Second Language Teacher who works in the Barnegat Township Public School District. Lynn is Certified at the State Level to train classrooms teachers in using the SIOP Model.

A Visual Approach to Solving Equations and Inequalities involving Absolute Value

Aradhana Kumari
Borough of Manhattan Community College
City University Of New York
akumari@bmcc.cuny.edu

Abstract: Solving absolute value equations and inequalities requires students to memorize a series of steps and follow these steps blindly. In my college algebra class, I have found that students have difficulty solving problems which contain absolute values. A visual approach using the number line and a "box" help students to solve problems which involve absolute values. In this article, we are emphasizing on breaking the problem into parts. We first make a picture of the problem using the number line and a "box" and write down the picture in mathematical words without using the absolute value symbol. We then solve the problem.

In this article, I present a visual method using the number line and box technique to solve equations and inequalities involving absolute values. Usually, a book begins a lesson on how to solve inequalities involving absolute values as follows: Let c be a positive real number

Equations and inequalities with absolute value symbol	Meaning without using absolute value symbol
$\lvert x\rvert = c$	$x = c \ \ or \ \ x = -c$
$\lvert x\rvert < c$	$-c < x < c$
$\lvert x\rvert \le c$	$-c \le x \le c$
$\lvert x\rvert > c$	$x < -c \ \ or \ \ x > c$
$\lvert x\rvert \ge c$	$x \le -c \ \ or \ x \ge c$

Table 1.

Students try to memorize the above table without understanding, and hence they make mistakes. Using the number line sweeps away the mystery of working with absolute values and empowers students to make connections between procedures and concepts (Mark W. Ellis and Janet L. Bryson, 2011).

Meaning of the absolute value symbol

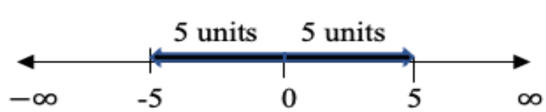
What is the meaning of symbol $|x|$?
$|x|$ means the directed distance of x from zero. Examples: |5| = 5, |-7| = 7 and |0|= 0.

Working with absolute value equations

Consider the following problem:

Problem 1: Solve for x, where $|x| = 5$.

Solution: This problem translates into finding all real numbers whose distance from the origin is 5 units. We draw the number line and find the numbers whose distance from the origin is five units.
The numbers are 5 or -5,



hence $x = 5 \ or \ x = \ -5$. The solution set is $x = \{-5, \ \ 5\}$.

Problem 2: Solve for x where $|x - 3| = 5$

Solution: Let us replace x-3 as a box$\square$.
Then our problem becomes
$|\square \ \ | = 5$
(1)

We saw from Problem 1 that equation given by (1) becomes
□ = 5 or □ = -5
(2)

We replace the box by x-3 in the equation given by (2), and obtain
$x - 3 = 5$ *or* $x - 3 = -5$
$x = 5 + 3$ *or* $x = -5 + 3$
$x = 8$ *or* $x = -2$.
Hence the solution set is $x = \{-2, 8\}$.

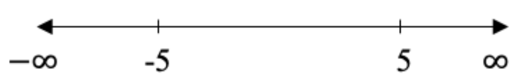
In summary, in the equations involving absolute value, expressions inside the absolute value can be thought of as a box, then our problem beco□es | |= c, where c is a non-negative real number. This problem can be translat□ as = c or □ = −c, and then we solve the problem for the given unknown variable which appears inside the box. As we saw in the Problem 2, the box could be any algebraic expression like x or $x - 3$.

Working with absolute value inequalities

Consider the following problem:

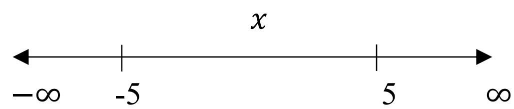
Problem 3: Solve for x, where
$$|x| < 5$$

Solution: This problem translates into finding all real numbers whose absolute value is less than 5.
Step 1: We draw the number line and label the numbers -5 and 5 on the number line.



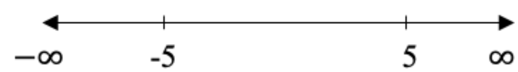
-5 and 5 on the number line divides the number line into three pieces; reading from left to right, the first piece is numbers less than -5, the second piece is numbers between -5 and 5, and the third piece is numbers bigger than 5.

Step 2: From each piece we take a number and check its absolute value. If the absolute value satisfies the inequality as given in Problem 3, then that piece is a solution. We take the number -6 from the first piece. Since |- 6 | = 6 and 6 is bigger than 5. Hence the first piece is not a solution. Next, we take the number 0 from the second piece. Since |0| = 0 and 0 is less than 5, the second piece is a solution. Lastly, we take the number 7 from the third piece. Since | 7 | = 7 and 7 is bigger than 5, the third piece is not a solution. Hence the solution is the second piece $-5 < x < 5$; that is numbers between -5 and 5.



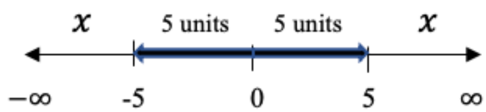
Problem 4: Solve for x, where
$$|x| > 5$$

Solution: This problem translates as find all real numbers whose absolute value is greater than 5.
Step1: We draw a number line and put the numbers -5 and 5 on the number line with



-5 and 5 divides the number line into three Pieces. Reading from left to right, the first piece is numbers less than -5, the second piece is numbers between -5 and 5, and the third piece is numbers bigger than 5.
Step 2: From each piece we take a number and check its absolute value. If the absolute value satisfies the inequality as given in Problem 4, then that piece satisfies the inequality and that piece is a solution. Let us take the number -6 from the first piece. Since |-6 | = 6 and 6 is bigger than 5, the first piece is a solution. We take the number 0 from the second piece, since | 0 | = 0 and 0 is less than 5, the second piece is

not a solution. Last, we take the number 8 from the third piece. Since | 8 | = 8 and 8 is bigger than 5, the third piece is a solution. Hence the solution is the first piece or the third piece; that is; numbers less than -5 or numbers bigger than 5, as shown below.



Problem 5: Solve for x, where

$$|3x + 2| > 5$$

Solution: We first construct the following problem, we replace 3x+2 by a box$\square$. Then the Problem 5 becomes

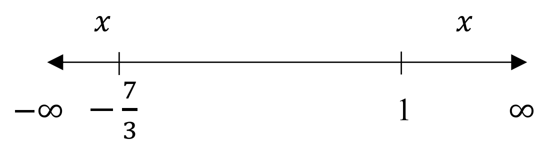
$$|\square\ | > 5 \qquad (3)$$

The problem given by inequality (3) is similar to Problem 4. The only difference is that we have box$\square$ inside the absolute value instead of x. From Problem 4 we saw that
the box$\square$ is less than 5; or the box$\square$ is greater than 5, that is

$$\square < -5 \quad \text{or} \quad \square > 5 \qquad (4)$$

Let us put the value of the box$\square$ in the inequalities given by (4), and we get

$$3x + 2 < -5 \quad \text{or} \quad 3x + 2 > 5$$
$$3x < -5 - 2 \quad or \quad 3x + 2 > 5$$
$$3x < -7 \quad or \quad 3x > 5 - 2$$
$$\frac{3x}{3} < -\frac{7}{3} \quad or \quad \frac{3x}{3} > \frac{3}{3}$$
$$x < -\frac{7}{3} \quad or \quad x > 1$$



Hence the solution is numbers less than $-\frac{7}{3}$ or numbers greater than 1.

Problem 6: Solve for x, where

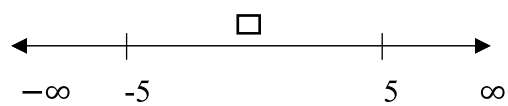
$$|2x + 3| < 5$$

Solution: We first construct the following problem. We replace 2x+3 by a box$\square$. Then the Problem 6 becomes

$$|\square\ | < 5 \qquad (5)$$

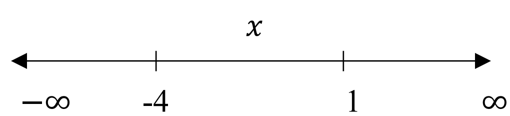
The problem given by inequality (5) is similar to Problem 3. The only difference is we have box$\square$ inside the absolute value instead of x. From Problem 3 we saw that the box$\square$ is in between -5 and 5, that is

$$-5 < \square < 5 \qquad (6)$$



Let us put the value of box in (6) so we get

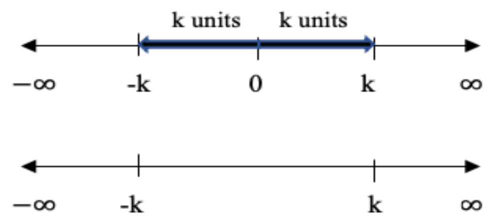
$$-5 < 2x + 3 < 5$$
$$-5 - 3 < 2x + 3 - 3 < 5 - 3$$
$$-8 < 2x < 2$$
$$\frac{-8}{2} < \frac{2x}{2} < \frac{2}{2}$$
$$-4 < x < 1$$



Summary

1. Considering Table 1., the main difficulty for students is that they do not understand how column 1 in the Table 1., is related to column 2. Hence, they try to memorize this table and commit errors. We can overcome this difficulty by putting one more column in the middle (as shown in Figure 1.) which uses the number line and the definition of absolute value. Figure 1., helps students understand the concept of absolute value, and it also helps them to see the relationship between column 1 and column 2 in Table 1.

2. We replace the algebraic expression inside the absolute value symbol with a box. We transform the problem so that the right--hand side of the inequality is a non-negative real number k. We then draw a number line and label k and -k on the number line (we draw k and -k because these are the two numbers whose distance from the origin is k units), k and -k divide the number line into three pieces as shown below. Reading from left to right, the first piece is numbers less than -k, the second piece is numbers between -k and k, the third piece is numbers bigger than k. We find (using the problem) on which piece(s) we keep the box; we write down the piece(s) using the math symbol inequality and the box. Lastly, we replace the box by the algebraic expression we started with and then we solve the inequality.



Conclusion

Presenting the lesson on how to solve equations and inequalities involving absolute values by the visual method using the box technique, we see that students do not need to memorize a series of steps; they understand the connection between the underlying concept and procedure on how to solve the problem which involves absolute value equations and inequalities. The box technique that is "replacing the complicated algebraic expression by a box" is a powerful technique and it can be used in other topics in mathematics like squaring of a number or squaring of an algebraic expression (Kumari. A, 2021).

Absolute Value Equations and Inequalities Interpretation

Absolute Value Equations and Inequalities:	Interpretation using the number line and box:	Interpretation without using absolute value symbol:

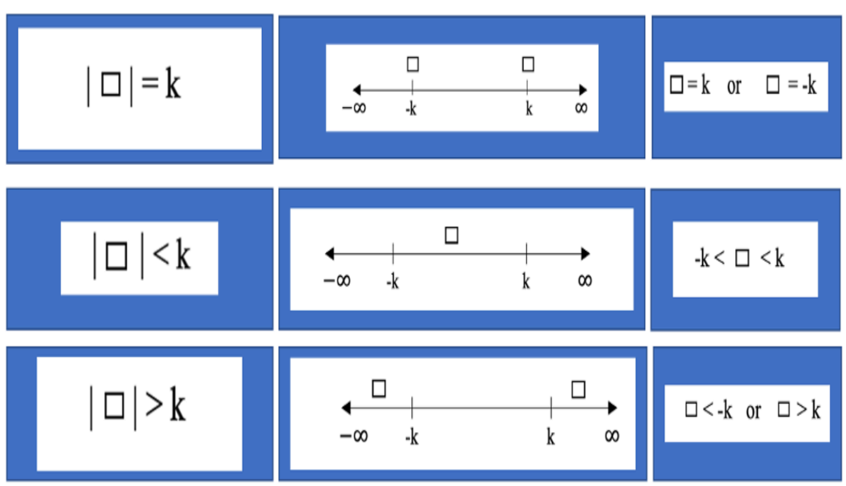


Figure 1.

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Dr. Aradhana Kumari received her Ph.D. in Mathematics from The Graduate Center, City University of New York, and is presently an Assistant Professor of Mathematics at the Borough of Manhattan Community College, City University of New York. Her research interest involves both mathematics and mathematics education. In mathematics, she is interested in three-dimensional Navier-Stokes equations. In mathematics education, she is interested in developing visual methods of teaching mathematical concepts. She is passionate about teaching Mathematics and believes that every student can learn mathematics if we provide proper instruction.

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Revisiting the Relationships between Multiplication with Fractions and Student Understanding

Leonard N. Masse
Carteret High School, Carteret, NJ

lmasse@carteretschools.org

Abstract: Students often encounter difficulty with fractions and some of this can be linked to a lack of understanding between the rules for fraction operations and the mathematical reasoning behind these rules. It is suggested here that an emphasis be placed on fraction multiplication as repeated addition and that the algorithms for multiplication be revealed through this process.

Introduction

Fractions seem often to be enemy of many students, even those who are good at problem-solving and who maintain a good sense of number in other areas of mathematics (Fennell 2007, Lortie-Forgues et al. 2015). There are many reasons for this and access to technology surely plays a role. A student is asked to divide 8 by 5 and it is very easy and quick to grab a calculator and see that the decimal equivalent of the answer is 1.6. It is rational for the student to behave in this manner. Decimal equivalents are seen often in everyday life and students generally understand their ordering and magnitude in the context of a problem or question. Although 8/5 is a perfectly acceptable answer in this case, it is one that is seldom provided. In this paper, I want to take some time to address how the connection between operations with fractions and answers is not well understood. The mechanics can be remembered and performed, but the reasoning behind the mechanics can remain a mystery for many of our students.

It has been a decade since the Common Core State Standards were first introduced with the call for deeper understanding and rigor in mathematics education. This paper will rely on these standards and the progression of multiplication with fractions beginning in grade five (CCSSI 2020). The approach and thoughts that follow are neither completely novel nor dramatic, but they bear repeating. *The emphasis will be on the basic notion of multiplication as repeated addition.* I have found the following approach and examples to be helpful in demystifying the mechanics behind multiplication with fractions.

Multiplication with Fractions – Repeated Addition

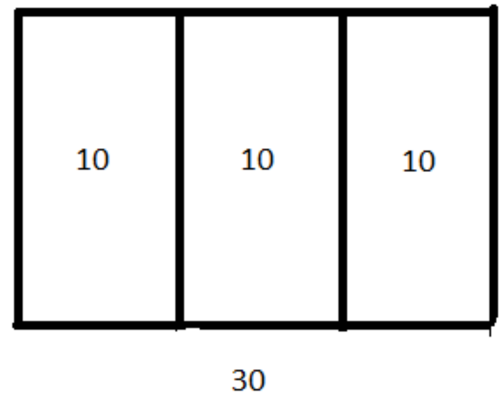
Let us start at the beginning. Why do we multiply when we want to find a fraction of a number? Let us consider how to find 3/5 of 20. We know that multiplication is repeated addition. What are we adding in this case? We know, and students would generally agree, that 3*20 is equal to 20+20+20 = 60. We can find the product of 3 and 20 using repeated addition and our rules for multiplication of whole numbers. Still, how does this relate to 3/5 of 20? What are we adding repeatedly? We can write that 3/5 is equivalent to 3*(1/5) and that 3/5 of 20 = 3(1/5)*20. We are adding (1/5)*20 three times. What is (1/5) of 20? We can employ our basic understanding of division of whole numbers and argue that (1/5) of 20 is 20/5 = 4. Therefore, we are adding 4 three times when we perform (3/5) of 20 and the answer is 12. The key here is to connect the process with one in which students are already familiar. Multiplication is repeated addition and it is helpful to find what is being added when fractions are involved.

How does the above procedure relate to the rules for multiplication with fractions? It does not relate very well. The rule is A/B *C/D = AC/BD. We have (3/5)*(20/1) and we used multiplication as repeated addition to help us to arrive at the answer and to understand the answer. We did not form the product AC or 60, and divide it by the product BD or 5. We found (A/D) or 3 and (C/B) or 4, and we considered this to be repeated addition of 4 three times to obtain our answer of 12. Children are first taught to consider multiplication in this manner and it is useful to consider multiplication of fractions to be a similar process.

How can the rule or algorithm be adjusted to fit our process of repeated addition in this case? We have:
(3/5) * (20/1) = 3 * (1/5) * 20 = 3*4 = 12. We found the part that we are adding three times and simply multiplied by 3. We know that (A/B) * (C/D) = (AC/BD) = (A/D) * (C/B) and we can generally consider this multiplication as repeated addition of (C/B) a total of (A/D) times.

Repeated Addition – Visual Representations and Examples

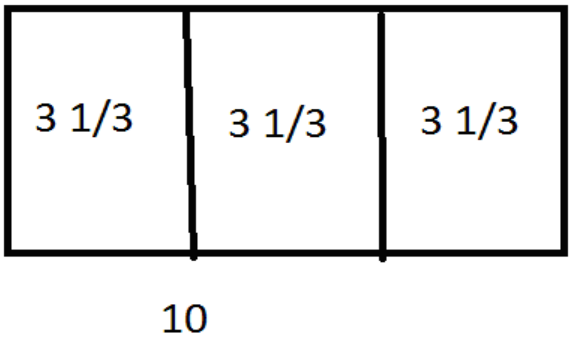
We will continue with the process that was presented above. What is 2/3 of 30? It is easy enough to follow the rule and multiply 2/3 and 30/1 and obtain 60/3 = 20.



It is easy to understand the above picture where the whole of 30 is divided into thirds of 10. Two of these thirds will give us 20. Again, this is repeated addition of the part (C/B = 30/3 = 10), a total of (A/D = 2/1 = 2) two times.

Let us explore again why the algorithm for multiplication gives us the correct answer? We have (2/1) * (30/3), which can be rewritten as (2/3) * (30/1). This is the setup that we use to multiply 2/3 and 30. The rule that we use hides the statement that helps us to more easily understand why the rule works. We have 2/3 of 30 is equal to 2*(30/3), which can be rewritten as (2/1)*(30/3) = 2*10 = 20. We have repeated addition of ten a total of two times. I think that this is an important, even vital, point in helping to improve our understanding of operations with fractions. The rules work but the reasons are concealed. Let us strip off the covering and reveal the truth hidden in our algorithms.

Let us continue with this approach in our consideration of 2/3 of 10? We will start as before. We will take 10, the whole, and divide it into thirds. We are dividing two whole numbers but our answer will be a fraction, and it may seem unwise to use a fraction early in our process when we are seeking to learn about multiplication with fractions. However, we will proceed secure in the knowledge that our students know that division of two whole numbers will not always result in a whole number.



We start out with a whole of 10 and we divide it into thirds. What is the size of each third? We have (10/3), which is equal to 3 1/3. We will leave it is as 10/3 for now. How many of these thirds do we need to find for this question? How many are we repeatedly adding in this case? We are adding two of them and therefore we need two of these thirds. Therefore, we have 2 ∗ (10/3), which is (2/1) ∗ (10/3), which can be rewritten as (2/3) ∗ (10/1). Again, the rule that we use hides the statement that helps us to more easily understand why the rule works. We have 2/3 of 10 is equal to (2/3) ∗ (10/1) by rule, which can be rewritten as (2/1) ∗ (10/3). The answer is 6 2/3, as we take two of the thirds in our picture. We can also simplify (2/3) ∗ (10/1) or (2/1) ∗ (10/3) and obtain 20/3 or 6 2/3. We are not abandoning the rule or process of multiplication with fractions. We are trying to unveil why it works.

Repeated Addition and Multiplication of Two Fractions

We now consider the case of multiplication and two fractions. We seek to relate the rule A/B ∗ C/D = AC/BD to our conceptual understanding of multiplication and fractions. We will examine the expression 4/5 ∗ 7/8.

We know that we are taking 4/5 of 7/8. We are taking a part of a fraction. We are taking 4 out of every 5 parts of 7/8. Let us represent 8/8 as eight Bs. Each B, therefore, has a value of 1/8.

B B B B
B B B B

We can illustrate 7/8 as 7 of the above Bs.

B B B B
B B B

We are taking 4/5 or 4 out of every 5 parts for the Bs. Again, we are using our understanding that a fraction N/D of X is asking us to take N out of every D parts of the number X.

How can we do this? We know that the rule tells us the 4/5 of 7/8 = (4∗5)/(7∗8) = 28/40 = 7/10.

B B B B
B B B

If we are taking four out of every five parts, then if seems logical to divide each B into five parts. Remember that a B has a value of 1/8 and our smaller parts will have a value of 1/40. We will represent these parts by b.

B B B B
B B B

bbbbb bbbbb bbbbb bbbbb

bbbbb bbbbb bbbbb

This is our representation of 7/8. We now need to perform 4/5 ∗ 7/8 by taking 4 out of every 5 parts of 7/8. We will take 4 bs out of every set of 5 bs. We will proceed by taking the four bs from each set of five. We can see that we have taken 4 ∗ 7 = 28 bs.

bbbb   bbbb   bbbb   bbbb

bbbb   bbbb   bbbb

What is the value of each b? We know that the value is 1/40. What do we have so far? We have $(4*7) * (1/40)$ or $(4 * 7)/40$. Let us recall where the 40 came from in this question. We had parts of 1/8 and we had to divide them further by 5 to obtain parts of 1/40. We obtained 40 smaller parts in the process, simply $5 * 8$.

Well, we can see that the answer is 28/40 = 7/10. Visually, we can rearrange the picture to see the answer is 7/10.

bbbb   bbbb   bbbb   bbbb
bbbb   bbbb   bbbb
bbbbbbb

We include the 7 bs that were not taken in the last row. We arrange the bs.

bbbb bbbb bbbb bbbb bbbb bbbb bbbb
bbbbbbb

Each b has a value of 1/40 and therefore each set of four bs has a value of 4/40 = 1/10. There are seven sets of four bs and the answer is 7/10. The seven bs in the last row represent what is left after we take 4/5 of 7/8. They represent 1/5 of 7/8. They are the 7 bs, each with a value of 1/40, that were not taken. They have a combined value of 7/40.

We need to relate what we have just done to the algorithm for multiplying fractions. The algorithm or rule is admittedly straightforward and "easy" relative to the rules for the addition and subtraction of fractions. Again, we are not abandoning the rule for an improved method. We are trying to uncover why the rule works and how it is related to our basic understanding of multiplication as repeated addition.

The rule tells us the 4/5 of 7/8 is equal to $(4 * 5)/(7 * 8) = 28/40 = 7/10$. We are taking 4/5 of 7/8. We present this as taking 4 out of every 5 parts of 7/8. The original parts of 7/8 are eighths. We divide the eighths into smaller parts so that we can take 4 out of every 5 of these smaller parts. The easiest way to do this is to simply divide each part by 5 so that we now have $8 * 5 = 40$ of the smaller parts. This is what we done so far to find 4/5 of 7/8:

$4/5 * 7/8 = 4 *$ (value of the smallest part) $*$ (number of sets or parts) =
$(4) * (1/40) * 7 = 4 * 7 * (1/40) =$ (total number of the smallest parts) $*$ (value of the smallest part).

We can see that this is really the repeated addition of 1/40 a total of 28 times. What does this mean for the rule in this case?

$4/5 * 7/8 = (4*7)/(5*8)$
$4/5 * 7/8 = (4*7) * (1/40)$

How does this relate to the general rule and to our notion of repeated addition? We will consider the repeated addition issue first. We can see that we found the smallest part (1/40) and we have repeatedly added it 28 times.

What about the rule: $A/B * C/D = AC/BD$? In our example, we can write that $4/5 * 7/8$ is equal to $(4 * 7) * (1/40)$. This is consistent with our previous notion of repeated addition. However, we found the smallest part first (1/40) and then we took 4 of these parts out of 7 sets. We have that $4/5 * 7/8 = (1/40) * 4 * 7$. What is the revised general rule using this method?
$A/B * C/D = (1/BD) * (AC)$

We have the repeated addition of (1/BD) at total of AC times. We use the commutative property of multiplication and we get the standard rule:

A/B $*$ C/D = AC / BD.
Again, our process allows the standard rule to be generated with an initial emphasis on multiplication as repeated addition

Concluding Comments

There are other ways to demonstrate the relationship between multiplication with fractions and the rule for fraction multiplication. The area model is one example that occurs often in texts and other instructional materials (Ervin 2017, PBS 2014). In some ways, the above method is similar in that you can consider the second factor in our examples to be the area of a figure in the plane. The product is the multiple or a fraction of the area that we are finding in the question. These are useful visuals. However, the primary goal of this paper was to shed some light on the rule for multiplication and why the rule works.

This process can be extended to the rules for division with fractions in a relatively straight-forward manner. Demystifying the rules for multiplication and division with fractions takes some effort but the returns may be well worth it in terms of student understanding.

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Dr. Leonard Masse is a graduate of Rutgers University, obtaining his B.S., M.S., and Ed.D. from that institution. He is presently a mathematics teacher at Carteret High School in Carteret, NJ, and is interested in the economics of education. He is a former research fellow at the National Institute of Early Education. He is also a bowling coach at the high school and had the distinction of bowling his second perfect game.

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