Answers

1. 5	**2.** $-\frac{1}{14}$	**3.** $-\frac{1}{2}$
4. 8	**5.** $2+\sqrt{7}$	**6.** 900
7. $50 + 10\pi$	**8.** 174.572	**9.** 0.582 or 0.581

Solutions

1. What is the units digit of the product $(2^2-1)(2^4-1)(2^6-1)\ldots.(2^{2022}-1)$?

Solution: This is the product of odd integers $3 \times 15 \times 63 \times 255 \times \ldots \times (2^{2022}-1)$ that contain multiples of 5. Therefore, the units digit must be 5.

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2. Given points $S(1,7), U(x,4)$ and $M(-4,-7)$.
Find <u>the exact value</u> of x that will minimize the sum $SU + UM$.

Solution: For the sum to be minimum, the three points must be collinear. To find *x*, find the equation of the line using *S* and *M,* then find the *x*-coordinate of *U* with *y*-coordinate 4.
Slope of the line is $\frac{14}{5}$, and its equation is $y - 7 = \frac{14}{5}(x-1)$.
By solving for *x* when *y* = 4, we find $x = -\frac{1}{14}$.

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3. Let $f(x) = 2x^3 + 3x^2 + cx + d$ where c and d are constants.
If 2 and –3 are real zeros of *f,* find the third zero.

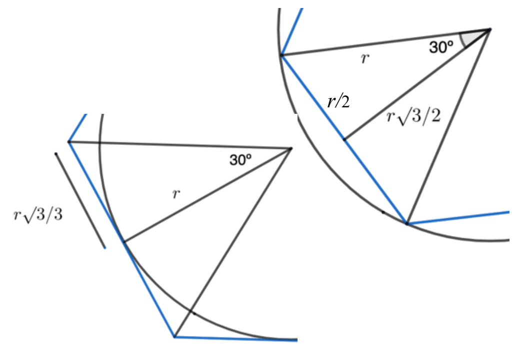
Solution: If 2 and –3 are zeros of *f,* then $f(2) = f(-3) = 0$
$f(2) = 0 \Rightarrow 28 + 2c + d = 0$ $\quad and\ f(-3) = 0 \Rightarrow -27 - 3c + d = 0.$
Solving the system of two equations, $c = -11$ and $d = -6$.
$\therefore f(x) = 2x^3 + 3x^2 - 11x - 6 = (x-2)(x+3)(2x+1)$, and the third zero is $-\frac{1}{2}$

4. A circle with radius r along with two regular hexagons are shown. One hexagon is inscribed in the circle and the other hexagon circumscribes the circle.
   Find the radius of the circle for which the sum of the areas of the two hexagons is $224\sqrt{3}\ cm^3$.

Solution: $A_i = Area_{inscribed\ hex} = 6 \times \frac{1}{2}\frac{r\sqrt{3}}{2} \times r = \frac{3}{2}r^2\sqrt{3}$, and

$$A_c = Area_{circumscribed\ hex} = 6 \times \frac{1}{2}r \times 2r\frac{\sqrt{3}}{3} = 2r^2\sqrt{3}$$

$$A_i + A_c = 224\sqrt{3} \implies \frac{7}{2}r^2\sqrt{3} = 224\sqrt{3} \implies r = 8\ cm.$$



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5. Find <u>the exact value</u> of the continued fraction $4 + \cfrac{3}{4 + \cfrac{3}{4+\cfrac{3}{4+\cfrac{3}{4+\dots}}}}$

Solution: Let $x = 4 + \cfrac{3}{4 + \cfrac{3}{4+\cfrac{3}{4+\cfrac{3}{4+\dots}}}}$. Then, $x > 0$ and $x - 4 = \frac{3}{x} \Rightarrow x^2 - 4x - 3 = 0 \Rightarrow x = 2 + \sqrt{7}$.

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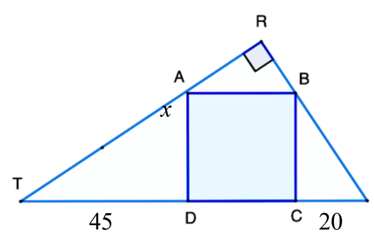
6. The square $ABCD$ is inscribed in right triangle TRI. Side $\overline{DC}$ is along the hypothenuse $\overline{TI}$, vertex A is along $\overline{TR}$, and vertex B is along $\overline{RI}$, as shown in the figure.
   If $TD = 45$ and $CI = 20$, what is the area of square $ABCD$?

Solution: Triangles ARB, TDA and BCI are similar.
This means, $\frac{TD}{BC} = \frac{DA}{CI} \implies \frac{20}{x} = \frac{x}{45}$
∴ Area of the square = 900

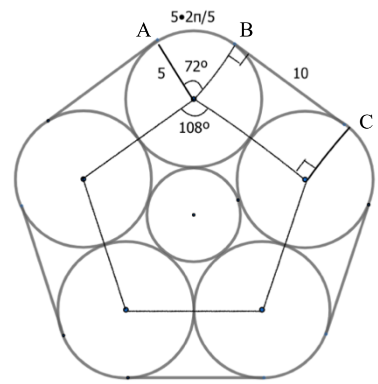


7. Five identical circles of radius 5 centimeters are arranged around a smaller circle, as shown in the figure. Each outer circle is tangent to two adjacent circles of radius 5 as well as the smaller circle in the center. The smaller circle in the center is tangent to all five larger circles. A band is tightly wrapped around the outside of the five circles. Find the exact length of the band.

Solution: The centers of the five circles are the vertices of a regular pentagon as shown.

$BC = 10cm$ because $\overline{BC}$ is tangent to the two circles.
The angle that subtends arcAB is 72º = $\frac{2\pi}{5}$ radians
$\Longrightarrow$ length of arc$AB = 5\left(\frac{2\pi}{5}\right) = 2\pi$.

∴ The length of the band is 5(10) +5(2π) = 50 + 10π cm.



8. At 20,310 feet (≈ 3.847 miles), Denali in Alaska is the highest mountain in North America. Given that the radius of the Earth is about 3,959 miles, find the distance (in miles) from the summit of Dinali to the furthest visible point along the horizon?

Solution: The adjacent figure shows a more complete (but unscaled) representation of Earth and Dinali,

$\Delta ACB \sim \Delta ABD \Longrightarrow \frac{AC}{AB} = \frac{AB}{AD} \Longrightarrow AB^2 = AC \bullet AD$
$\Longrightarrow AB^2 = 3.847(3.847 + 2 \bullet 3959) = 30475.345$

∴ The distance from the summit to the furthest visible point along the horizon is $\sqrt{30475.345} = 174.572$ miles.

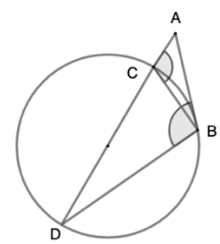


Figure is not drawn to scale

9. The table below shows the percentage of party affiliations among all registered voters in a local municipality and the percentage within each party who voted during a recent local election.

	Party A	Party B	Party C
Party Affiliation	30%	50%	20%
% Who Voted	65%	82%	50%

Given that a randomly selected person from the municipality has voted in the election, what is the probability that she is a member of party B?

Solution: Let *N* be the total number of registered voters in the local municipality. Here is the table of counts that corresponds to the table of percentages:

	Party A	Party B	Party C	Totals
Numbers in each party	.3*N*	.5*N*	.2*N*	*N*
Numbers who voted	.6(.3*N*)	.82(.5*N*)	.5(.2*N*)	.705*N*

∴ The probability that the random person who voted was a party B member is $\frac{.82(.5)N}{.705N} = .582$