Answers

1. 32

2. 2023

3. 161

4. $-\frac{71}{12}$ or $5\frac{11}{12}$ or $-5.91666\ldots$ or $-5.91\overline{6}$

5. $\sqrt{6}$

6. $\frac{21\sqrt{3}}{4}$ or 9.093

7. 37,950

8. $13\sqrt{3}-\frac{13}{3}\pi$ or 8.903

9. 23

Solutions

1. Let $f(x)=x^2+ax+b$. If $f(-3)+f(3)=0$, find $f(-5)+f(5)$.

Solution: $f(-3)+f(3)=9-3a+b+9-3a+b=0 \Rightarrow 18+2b=0 \Rightarrow b=-9$
$\therefore f(-5)+f(5)=25-5a-9+25+5a-9=32$

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2. If the longest side of a right triangle is $10^{2023}+1$ units, and the other sides are $10^{2023}-1$ units and $n10^m$ units, find the value of $n\times m$.

Solution: The third side = $\sqrt{(10^{2023}+1)^2-(10^{2023}-1)^2}=\sqrt{2\cdot 2\cdot 10^{2023}}=2\cdot 10^{\frac{2023}{2}} \Longrightarrow n\times m=2023$

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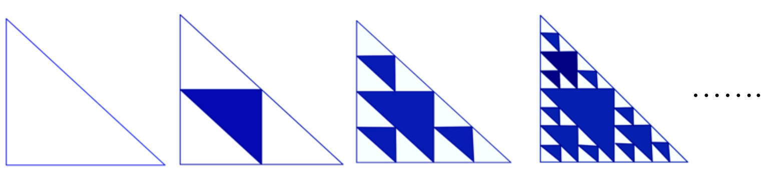
3. In the sequence of triangles shown below, stage 0 has one triangle and stage 1 has five triangles. If the pattern continues, how many triangles will stage 4 have?



stage 0 stage 1 stage 2 stage 3

Solution 1:

Stages	stage 0	stage 1	stage 2	stage 3	stage 4
Number of triangles	1	1+ 4 = 5	5 + 4(3) = 17	17 + 4(9) = 53	53 + 4(27) = 161

Solution 2:

Stage n	Number of triangles = $2(3^n)-1$
stage 0	1
stage 1	1 + 3 + 1 = 5
stage 2	1 + 3 + 9 + 3 + 1 = 17
stage 3	1 + 3 + 9 + 27 + 9 + 3 + 1 = 53
stage 4	1 + 3 + 9 + 27 + 81 + 27 + 9 + 3 + 1 = 161

4. A function f has zeros at 3, $\frac{5}{8}$ and $-\frac{2}{3}$. If $g(x) = -3f\left(-\frac{x}{2}\right)$, what is the sum of the zeros of g?

Write your answer in exact form.

Solution: g is obtained by stretching f horizontally by a factor of 2, reflecting it in the y-axis, stretching it vertically by a factor of 3, then reflecting over the x-axis. Only the first two transformations have an effect on the zeros.

When f is reflected and stretched horizontally, 3, 5/8, and –2/3 shift to –6, –5/4 and 4/3 respectively, and their sum $= -6 - \frac{5}{4} + \frac{4}{3} = -\frac{71}{12}$ or $5\frac{11}{12}$ or the repeating decimal $-5.91666\ldots$ or $-5.91\overline{6}$

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5. Water in a large cylindrical tank is 100 inches deep. When a cylinder with a smaller base is placed in the tank, the water level rises to 120 inches, as shown on the right.

If the large tank has radius R, and the smaller cylinder has radius r, find the <u>exact value</u> of $\frac{R}{r}$.

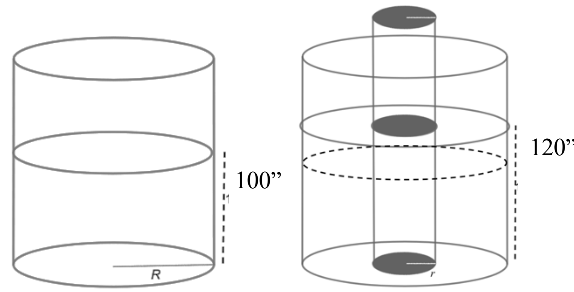

Figure is not drawn to scale

Solution: $100\pi R^2 = 120\pi R^2 - 120\pi r^2$ because the volume of water does not change.

$\Rightarrow 10R^2 = 12R^2 - 12r^2 \;\Rightarrow R^2 = 6r^2 \;\Rightarrow \frac{R}{r} = \sqrt{6}.$

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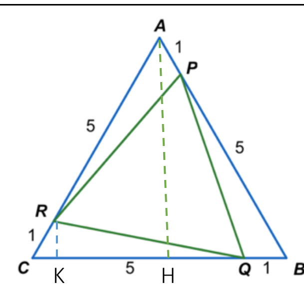
6. Equilateral triangle PQR is inside another equilateral triangle ABC, with P, Q and R along sides AB, BC and CA respectively, one unit away from each vertex as shown in the figure. If $AB = BC = CA = 6$, find the area of triangle PQR.

Solution: Area of triangle PQR = Area of triangle ABC – 3(Area of triangle CRQ)

$$= \frac{1}{2}AH \cdot CB - 3\left(\frac{1}{2}RK \cdot CQ\right)$$
$$= \frac{1}{2} \cdot 3\sqrt{3} \cdot 6 - 3 \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2} \cdot 5$$
$$= 9\sqrt{3} - \frac{15\sqrt{3}}{4} = \frac{21\sqrt{3}}{4}$$



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7. In a lottery game at the local fair, a player chooses four distinct numbers between 1 and 25 for the chance to win \$10,000. To win, your 4 numbers must match the 4 randomly chosen numbers at the drawing. Each play costs \$3, and you can play this game multiple times. What is the least amount you must spend to guarantee a win?

Play for \$3 Pick 4 for a chance to win \$10,000				
1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25

Solution: To guarantee a win, all combinations of 4 distinct numbers 1 through 25 must be played. This is $\frac{25\cdot24\cdot23\cdot22}{4!} = 12{,}650$. At \$3 per game, the least amount spent to guarantee a win would be \$37,950.

8. Two lines that intersect at I form 120º, and the two circles with radii 2 and 3 are tangent to the lines at points F, G, H and K as shown in the figure on the right.
What is the area of the shaded region?

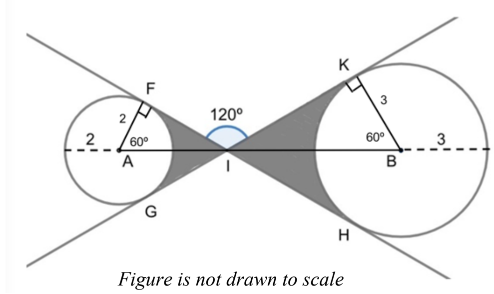

Figure is not drawn to scale

Solution: $\Delta\, IKB$ and $\Delta\, IFA$ are half equilateral triangles and their areas are $\frac{9\sqrt{3}}{2}$ and $\frac{4\sqrt{3}}{2}$ respectively.
The areas of circular sectors KBB' and FAA' are $\frac{3}{2}\pi$ and $\frac{2}{3}\pi$ respectively.
$\therefore$ The area of the shaded region = $(9\sqrt{3} - 3\pi) + (4\sqrt{3} - \frac{4}{3}\pi) = 13\sqrt{3} - \frac{13}{3}\pi \approx 8.903$

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9. The n^{th} term a_n of a sequence of numbers $a_1, a_2, a_3, \ldots$ is defined by $a_n = a_{n-a_{n-1}} + a_{n-a_{n-2}}$,
where $a_1 = 1$ and $a_2 = 2$.

Find $a_1 + a_2 + a_3 + \ldots + a_7$, the sum of the first 7 terms of this sequence.

Solution: *Given*: $a_1 = 1$ *and* $a_2 = 2$.

$a_3 = a_{3-a_2} + a_{3-a_1} = a_1 + a_2 = 3$

$a_4 = a_{4-a_3} + a_{4-a_2} = a_1 + a_2 = 3$

$a_5 = a_{5-a_4} + a_{5-a_3} = a_2 + a_2 = 4$

$a_6 = a_{6-a_5} + a_{6-a_4} = a_2 + a_3 = 5$

$a_7 = a_{7-a_6} + a_{7-a_5} = a_2 + a_3 = 5$

$\therefore\ a_1 + a_2 + \ldots + a_7 = 23$