**Answers:**

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**Problems:**

1. **39**
2. **166**
3. **1/10**
4. **32**
5. **25**
6. **8**
7. **1,537,734**
8. $96\sqrt{2}$ **or 135.764 or 135.765 square units.**
9. $\frac{1}{4}\pi$ **inches or 0.785 inches.**

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1. For how many different positive integers n does $\sqrt{n}$ differ from $\sqrt{100}$ by less than 1?

Answer: If the integer n is greater than $9^2 = 81$ but less than $11^2 = 121$, then $\sqrt{n}$ differs from 10 by less than 1. The set $\{82, 83, ..., 119, 120\}$ contains 39 different integers.

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2. The two roots of the quadratic equation $x^2 - 85x + c = 0$ are prime numbers. What is the value of c?

Answer: Assuming that two prime numbers $x_1$ and $x_2$ are the solutions, then $x_1 + x_2 = 85$. Since 85 is an odd number, either $x_1$ or $x_2$ must be an even number. The only even prime number is 2. Therefore, one number must be 2 and the other 83. Hence $c = 2\times83 = 166$.

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*3.* A class has three girls and three boys. These students line up at random, one after another. What is the probability that no boy is right next to another boy, and no girl is right next to another girl? Write your answer as a fraction in simplest form.

Answer: We pick the students in order. The first choice is always okay. There is a probability $\frac{3}{5}$ that the second choice is okay; for example, if you picked a boy first then there are two boys and three girls left. Similarly, the probabilities that the remaining choices are okay are $\frac{1}{2}$, $\frac{2}{3}$, $\frac{1}{2}$, 1. Find the product of all of these probabilities. $1\left(\frac{3}{5}\right)\left(\frac{1}{2}\right)\left(\frac{2}{3}\right)\left(\frac{1}{2}\right)(1) = \frac{1}{10}$

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4. Mary paid \$480 to purchase a certain number of items, but the nice vendor gave her two extra. This decreased the price per item by \$1. How many items did she receive (including the two extra)?

Answer: If x is the number of items she received, then

$$\frac{480}{x-2} - \frac{480}{x} = 1$$

$$480x - 480(x - 2) = x(x - 2)$$
$$480x - 480x - 960 = x^2 - 2x$$
$$x^2 - 2x + 960$$
$$(x + 30)(x - 32) = 0$$
$$x = 32 \text{ items}$$

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5. A ladder is leaning against a house with its bottom 15 feet from the house. If the bottom is pulled 9 feet farther away from the house, the upper end would slide 13 feet down. How many feet long is the ladder?

Answer: If L denotes the length of the ladder and H the height to which it extends initially, then

$$L^2 = H^2 + 225 \text{ and } L^2 = (H - 13)^2 + 24^2$$

$$H^2 + 225 = (H - 13)^2 + 24^2$$
$$H^2 + 225 = H^2 - 26H + 169 + 576$$
$$26H = 520$$
$$H = 20 \text{ feet}$$
$$L = 25 \text{ feet}$$

6. You have two boxes. Each of them has a square base and is half as tall as it is wide. If the larger box is two inches wider than the smaller box, and has a volume 244 $in^3$ greater, what is the width of the smaller box?

Answer: If w is the width of the smaller box, then the volume of this box is $\frac{w^3}{2}$ and the volume of the larger box is $\frac{(w+2)^3}{2}$. We get

$$\frac{(w+2)^3}{2} - \frac{w^3}{2} = 244$$
$$(w + 2)^3 - w^3 = 488$$
$$6w^2 + 12w + 8 = 488$$
$$6w^2 + 12w - 480 = 0$$
$$6\left(w^2 + 2w - 80\right) = 0$$
$$6(w + 10)(w - 8) = 0$$

Since w needs to be positive, the width of the smaller box must be 8 inches.

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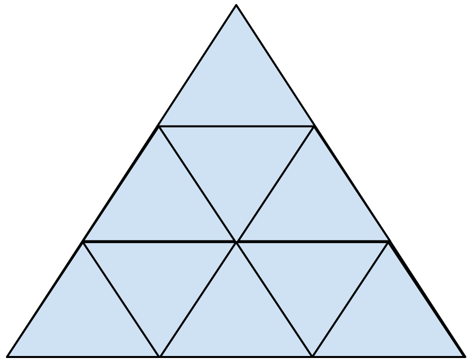
7. A large equilateral triangle is constructed by using toothpicks to create rows of small equilateral triangles. For example, in the figure we have 3 rows of small congruent equilateral triangles, with 5 small triangles in the base row. How many toothpicks would be needed if the base row of the triangle consists of 2023 small equilateral triangles?



| # of triangles | Sum of toothpicks |
|---|---|
| $a_1 = 1$ | $S_1 = 3$ |
| $a_2 = 3$ | $S_2 = 3 + 6 = 3(1 + 2)$ |
| $a_3 = 5$ | $S_3 = 3 + 6 + 9 = 3(1 + 2 + 3)$ |

$a_n = 2n - 1$ (odd sequence)

$a_n = 2023$ Since the base row consists of 2023 triangles, the base row is calculated to be row 1012

$2023 = 2n - 1$
$2024 = 2n$
$1012 = n$

The sum of all triangles in the entire triangle are found using an arithmetic series computation:
$S_{1012} = 3\ (1 + 2 + 3 + \ldots + 1012) = 3\ (1 + 1012)(\frac{1012}{2}) = 1,537,734$

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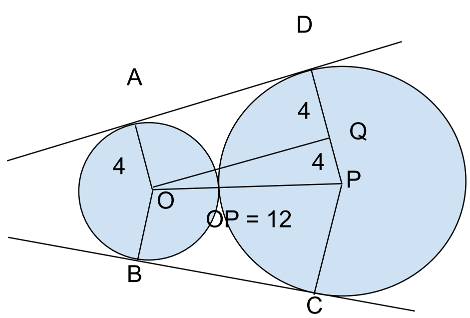
8. Circles with Centers O and P have radii of 2 and 4, respectively, and are externally tangent. Points A and B are on the circle centered at O, and points C and D are on the circle centered at P, such that $\overline{AD}$ and $\overline{BC}$ are common external tangents to the circles. What is the area of the hexagon AOBCPD?




Tangency of AD to both circles creates right angles between $\overline{AD}$ and $\overline{AO}$ and $\overline{DP}$, respectively.
$\overline{OQ} = \sqrt{OP^2 - 4^2} = \sqrt{128} = 8\sqrt{2}$

The area of OADP is $\left(4 \cdot 8\sqrt{2}\right) + \left(\frac{4 \cdot 8\sqrt{2}}{2}\right) = 48\sqrt{2}$
The area of AOBCPD is $96\sqrt{2}$ or 135.764 or 135.765 square units.

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9. A rectangular fish tank has the base dimensions 4 feet by 2 feet, and height 3 feet. The tank is initially half full. A solid steel ball of diameter 1 foot is dropped into the tank, which sinks to the bottom. By approximately how many inches will the tank rise? Note: 1 foot = 12 inches. (Practice exam 8 #4)

The radius of the steel ball is .5 feet, so it's volume is $\frac{4}{3}\pi(\frac{1}{2})^3 = \frac{1}{6}\pi$ cubic feet. If the volume of water in the tank is V cubic feet, then the volume of water and the steel ball is $V + \frac{1}{6}\pi$ cubic feet. The height (water level) is obtained by dividing the volume by the base area, and the difference in water level can be obtained by dividing the volume of the ball by the base area.

Since the base area is 8 $ft^2$, the change in water level will be $\frac{\frac{1}{6}\pi\, ft^3}{8ft^2} = \frac{1}{48}\pi$ ft. Convert feet to inches by multiplying by 12. Water level changes by $\frac{1}{4}\pi$ inches or 0.785 inches.