Answers:

Problems:

1. **18**
2. **606**
3. **90**
4. **114 and 1560**
5. **55**
6. **7/2 and 4**
7. **⅔ or 2:3**
8. **-4/3.**
9. **(10,90) and (12,36)**

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1. If a<b, then $3^2+4^2+5^2+12^2=a^2+b^2$is satisfied by only one pair of positive integers (a,b). what is the value of ab?

Answer: $3^2+4^2 = 5^2$ *and* $5^2+12^2=13^2 \ldots so \ldots a = 5$ *and* $b = 13; \; a + b = 18$

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2. This coming Halloween, Tom plans to scare twice as many people as Sam, and Sam plans to scare three times as many people as Roz. In all, they plan to scare at most 2025 people. If no one is scared more than once, at most, how many people does Sam plan to scare?

Answer: If Roz plans to scare n people, then Sam plans to scare 3n people and Tom plans to scare double that, 6n. Altogether, n + 3n + 6n $\leq$ 2025; so n $\leq$ 202.5. Since n is an integer, n is at most 202. Since Sam plans to scare 3n people, that's at most 606 people.

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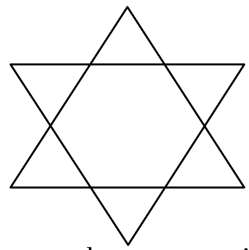
3. When two congruent equilateral triangles share a common center, their union can be a star, as shown. If their overlap is a regular hexagon with an area of 60, what is the area of one of the original equilateral triangles?



Answer:
Method 1: Draw the 3 diagonals of the hexagon, as shown, to partition the figure into 12 congruent small equilateral triangles. Since the overlap consists of 6 of these triangles with a total area of 60, each small triangle has an area of 10. Since each original (large) triangle consists of 9 small ones, the area of a large equilateral triangle is 90.

Method 2: Join the common center to any two consecutive vertices of the hexagon. The equilateral triangle inside the hexagon is congruent to each equilateral triangle outside the hexagon. Since six of these "inside" triangles make up the hexagon, a large equilateral triangle consists of these six triangles plus 3 additional triangles. Therefore, the area of a large equilateral triangle is 60 + (½)60 = 90.

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4. Between 1 and 200, there is a sequence of 13 consecutive integers, none of which is prime. What is the sum of these integers, and what is the smallest of these integers?

Answer:

We could list all prime numbers under 200, but since we seek a gap of 13 (or more) between primes, so let us list selected primes instead. For as long as possible, try to leave a gap of no more than 12 between listed primes. In the list 11,23,31,41,53,61,71,79,89,97,103,113, the difference between two primes adjacent on the list never exceeds 12. Hens the first of 13 consecutive composites exceeds 113. In fact, the next prime is 127. Here's why: 115,117,119,121,123 and 125 are respectively divisible by 5,3,7,11,3 and 5; so 114,115, … and 126 are all composite. The sum of these integers is 1560, and the smallest of these integers is 13.

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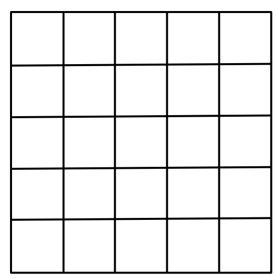
5. Counting every possible square of each size from 1x1 to 5x5, what is the total number of distinct squares that can be traced out along the lines of the accompanying grid?



Answer:

Each 1x1 square is determined by its upper left vertex, for which there are 5x5 choices. Each 2x2 square is determined by its upper left vertex, for which there are 4x4 choices. Each 3x3 square is determined by its upper left vertex, for which there are 3x3 choices. In general, each dxd square is determined by its upper left vertex, for which there are $(6-d)(6-d) = (6-d)^2$ choices. Thus the number of 1x1 squares is 5^2, the number of 2x2 squares is 4^2, …, and the number of 5x5 squares is 1^2. The total number of squares of all sizes is $5^2+4^2+3^2+2^2+1^2 = 55$.

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6. The four numbers $a<b<c<d$ can be paired in six different ways. If each pair has a different sum, and if the four smallest sums are 1,2,3, and 4, what are all possible values of d?

Answer:

The six possible sums are a+b, a+c, b+c, a+d, b+d, and c+d. Since $a < b < c < d$, the smallest sums are a+b = 1, a+c = 2. From these, c = b+1. Of the other four sums, the largest are b+d and c+d. Of the remaining sums, b+c and a+d, one equals 3, and the other equals 4. Since c = b+1, if b+c = 2b+1 = 3, then b = 1. Then since a+b = 1, a = 0; and since a + d = 4, d = 4. If instead, b + c = 2b+1 = 4, then b = 3/2. Then, since a + b = 1, a = -½; and since a+d = 3, d = 7/2. Finally, the two possible values of d are 7/2 and 4.

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7. Let ℓ_1 be the perimeter of the ellipse $\frac{x^2}{36}+\frac{y^2}{16}=1$ and ℓ_2 be the perimeter of the ellipse $\frac{x^2}{36}+\frac{y^2}{81}=1$. What is the ratio of ℓ_1 to ℓ_2

Answer:

Method 1: The first ellipse has its major axis on the x-axis. The length of its major axis is 12 and the length of the minor axis is 8. The second ellipse has its major axis on the y-axis. Its major axis has a length of 18 and its minor axis is 12. The ratio of the lengths of each set of axes is 2:3, so the ellipses are similar, with a ratio of similitude of 2:3.

Method 2: The larger (second) ellipse is obtained from the smaller (first) ellipse by the competition of a 90° rotation centered at the origin, with a 150% dilation also centered at the origin. A 150% dilation of ellipse 1 to create ellipse 2 would result in a ratio of ℓ_1 to ℓ_2 to be 2:3.

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8. If (a,2025) and (2025,b) are different points on the graph of the line $y=\frac{3}{4}x+5$, what is the value of $\frac{2025-a}{2025-b}$?

Answer:

The slope of the line is $\frac{b-2025}{2025-a}$, which equals $\frac{3}{4}$. We're asked for $\frac{2025-a}{2025-b}$, which is the negative reciprocal of the given slope. Its value is - 4/3.

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9. What are all the ordered pairs of (a,b), with 0 < a < b, for which $(\sqrt[a]{2025})(\sqrt[b]{2025})=\sqrt[9]{2025}$?

Answer:

$\left(2025^{\frac{1}{a}}\right)\left(2025^{\frac{1}{b}}\right)=2025^{\frac{1}{a}+\frac{1}{b}}$, so we must find integer solutions to $\frac{1}{a}+\frac{1}{b}=\frac{1}{9}$, with 0 < a < b

Method 1: Since a< b, it follows that 9 < a< 18. By trial of the 8 possible values of a, we find that a = 10 or 12, so (a,b) = (10,90), (12,36)

Method 2: Clearly, a > 9 and b > 9. Let a = 9+x and let b = 9 + y, where x and y are positive integers with 0 < x < y. After substituting, clear the resulting equation of fractions to get xy = 81. Finally, (x,y) = (1,81), (3,27), so (a,b) = (10,90), (12,36)
