Answers

Problems:

1. **-18**
2. **2026**
3. **6**
4. **22**
5. **4π**
6. **(a,b) = (19,18), (8,5) and (7,3)**
7. $(x^2 + 2x + 2)(x^2 - 2x + 2)$
8. **$8.49**
9. **49¢**

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1. If a is an integer, and if both roots $x^2 + ax + 17 = 0$ are positive integers, what is the value of a?

Answer: Since 17 is prime, any positive integral root comes from (x-17)(x-1) = $x^2 - 18x + 17, \; so\; a = -18$

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2. What is the minimum value of |x + y + z |, given that |x| = 2027, |y| = 2028 and |z| = 2029.

Answer: The minimum value of $| \pm 2027 \pm 2028 \pm 2029|$ *is* |2027 + 2028 - 2029| = |2029 - 2028 - 2027| = **2026**

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3. All five sides of a *Pythagorean Pentagon* have integral lengths, and the sum of the squares of the lengths of the four smallest sides is equal to the square of the length of the longest side. What is the least possible perimeter of a Pythagorean Pentagon?

Answer: Since $1^2 + 1^2 + 1^2 + 1^2 = 2^2$, the least possible perimeter is 1 + 1 + 1 + 1 + 2 = **6**

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4. For all real numbers x, the function f is periodic, with f(x + 6) = f(x + 10) = f(x) If f(22)= 22, what is the value for f(44)?

Answers: Since f(x) = f(x+6) = f(x + 6 + 6) = f(x + 2 + 10) = f(x + 2), we know that f(x) = f(x + 2) for all real x. Hence, f(22) = f(24)= f(26) … = f(44), so f(22) = f(44) = **22**

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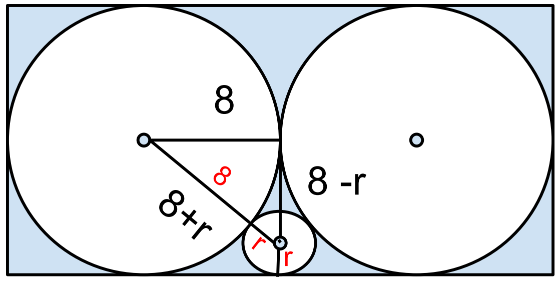
5. As shown below, two congruent circles are inscribed in a rectangle so that each is tangent to three sides of the rectangle and to the other circle. A third circle, smaller than the other two, is tangent to both congruent circles and one side of the rectangle. If the area of each congruent circle is 64π, what is the area of the small circle?

Answer: In the right triangle shown, the hypotenuse is 8 + r and the legs are 8 and 8 - r. By the Pythagorean theorem, $8^2 + (8 - r)^2 = (8 + r)^2$; so r = 2 and the area of the small circle is **4π**



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6. What are all the pairs of positive integers (a,b) for which $a^2 + b$ exceeds $a + b^2$ by 36?

Answer -

$(a^2 + b) - (a + b^2) = 36$
$(a^2 - b^2) - (a - b) = 36$
$(a + b)(a - b) - (a - b) = 36$
$(a - b)(a + b - 1) = 36$

Since a and b are positive integers, the largest factor (a + b -1) is a positive integer. Thus, the factor (a-b) is also positive. One of these factors must be even; the other must be odd. Factoring numerically, we get

a + b - 1 = 36 12 9
a - b = 1 3 4

So 2a - 1 = 37 15 13

And a = 19, 8, 7

Finally (a,b) = (19,18), (8,5) and (7,3)

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7. Factor $x^4 + 4$; that is, write $x^4 + 4$ as a product of two quadratic polynomials with integral coefficients.

Answer:

$$x^4 + 4 = (x^4 + 4x^2 + 4) - (4x^2)$$
$$= (x^2 + 2)^2 - (2x)^2$$
$$= (x^2 + 2 + 2x)(x^2 + 2 - 2x)$$
$$= (x^2 + 2x + 2)(x^2 - 2x + 2)$$

8. A bag of 5 apples, 7 bananas and 3 carrots cost \$4.41; and a bag of 6 apples, 2 bananas and 1 carrot costs \$2.37. At these prices, how much should a bag of 3 apples, 17 bananas, and 7 carrots cost?

Answer:

Let's see how we might get the required 3 apples and other fruit from pre-filled bags of 5 apples and other fruit or 6 apples and other fruit. One way is to buy 3 five-apple bags and sell 2 six-apple bags to my friend. In fact, 3(5a + 7b + 3c = \$4.41) - 2(6a +2b +c = \$2.37) gives us the result 3a + 17b + 7c = \$8.49.

Note: this problem would have no solution unless the contents and price of the new collection of fruit was dependent on the two conditions in the first sentence of the problem. The dependency can be seen in the linear combination x(5a + 7b + 3c = \$4.41) + y(6a + 2b + c = \$2.37), from which a(5x + 6y) + b(7x + 2y) + c (3x + y) = \$4.41x + \$2.37y. We want the value of 3a + 17b + 7c, so we try 5x + 6y = 3, 7x +2y = 17, and 3x + y = 7. The system is consistent when x = 3 and y = -2. Finally \$4.41 x + \$2.37 y = \$13.23 - \$4.74 = \$8.49.

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9. The cost (in cents) of n candies is equal to the number of candies that I can buy for 98 cents. At the same cost per candy, how many cents do 14 candies cost?

Answer:

Method 1 - Let c be the cost of one candy, in ¢. Then $8c = \frac{98}{c}$, so $c^2 = 98/8 = 49/4$. Thus, c = 7/2, and 14c = 49 or 49 ¢.

Method 2 - If n candies can be bought for 98 ¢, then the number of cents that each candy costs is 98/n. But n is also the number of cents that it costs to buy 8 candies, so the cost of each, in cents is n/8. Equating 98/n = n/8, so n = $\sqrt{8 * 98} = 28$. Since 28 is the number of candies that can be bought for 98 ¢, the cost of 14 candies would be 49 ¢.